MANONMANIAM SUNDARANAR UNIVERSITY

TIRUNELVELI - 627012

DIRECTORATE OF DISTANCE AND
CONTINUING EDUCATION

M.Sc. MATHEMATICS

FIRST YEAR

ORDINARY DIFFERENTIAL EQUATIONS

Sub. Code: SMAM13

SMAM13: ORDINARY DIFFERENTIAL EQUATIONS

SYLLABUS

UNIT-I : Linear equations with constant coefficients: Second order homogeneous equations - Initial value problems - Linear dependence and independence - Wronskian and a formula for Wronskian - Non-homogeneous equation of order two.
(Chapter 2: Sections 1 to 6)

UNIT-II : Linear equations with constant coefficients: Homogeneous and non-homogeneous equation of order n - Initial value problems - Annihilator method to solve non-homogeneous equation - Algebra of constant coefficient operators.
(Chapter 2 : Sections 7 to 12)

UNIT-III : Linear equation with variable coefficients: Initial value problems - Existence and uniqueness theorems - Solutions of a homogeneous equation - Wronskian and linear independence - reduction of the order of a homogeneous equation - The non-homogeneous equation - homogeneous equation with analytic coefficients - The Legendre equation.
(Chapter 3 : Sections 1 to 8 (Omit section 9))

UNIT-IV : Linear equation with regular singular points: Euler equation - Second order equations with regular singular points - Exceptional cases - Bessel Function.
(Chapter 4 : Sections 1 to 4 and 6 to 8 (Omit sections 5 and 9))

UNIT-V : Existence and uniqueness of solutions to first order equations: Equation with variable separation - Exact equation - method of successive approximations - The Lipschitz condition - convergence of the successive approximations and the existence theorem.
(Chapter 5 : Sections 1 to 6 (Omit Sections 7 to 9))

Recommended Text: E.A.Coddington, *A Introduction to Ordinary Differential Equations* (3rd Printing) Prentice-Hall of India Ltd., New Delhi, 1987.

Contents

Chapter 1

Linear equations with constant coefficients

1.1 Introduction

A linear differential equation of order n with constant coefficients is an equation of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = b(x)$$

where $a_0 \neq 0, a_1, \cdots, a_n$ are complex constants and b is some complex valued function on an interval I. By dividing by a_0 we can arrive at an equation of the same form with a_0 replaced by 1. Therefore we can always assume $a_0 = 1$, and our equation becomes

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = b(x) \tag{1.1}$$

It will be convenient to denote the differential equation on the left of the equality (1.1) by $L(y)$. Thus

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y$$

and the equation (1.1) becomes simply $L(y) = b(x)$. If $b(x) = 0$ for all x in I the corresponding equation $L(y) = 0$ is called a *homogeneous equation*, whereas if $b(x) \neq 0$ for some x in I the corresponding equation $L(y) = b(x)$ is called a *non-homogeneous equation*. We give meaning to L itself as a *differential operator* which operates on function which have n derivatives on I, and transforms such a function

ϕ into a function $L(\phi)$ whose value at x is given by

$$L(\phi(x)) = \phi^{(n)}(x) + a_1\phi^{(n-1)}(x) + \cdots + a_{n-1}\phi'(x) + a_n\phi(x).$$

Thus

$$L(\phi) = \phi^{(n)} + a_1\phi^{(n-1)} + \cdots + a_{n-1}\phi' + a_n\phi.$$

A solution of $L(y) = b(x)$ is therefore a function ϕ having n derivatives on I such that $L(\phi) = b$. If b is continuous on I, then it is possible to find all solutions of $L(y) = b(x)$. In this chapter we consider the case of second order equation ($n = 2$).

1.2 Second order homogeneous equations

Here we are concerned with the equation

$$L(y) = y'' + a_1y' + a_2y = 0 \tag{1.2}$$

where a_1 and a_2 are constants. We recall that the first order equation with constant coefficients $y' + ay = 0$ has a solution e^{-ax}. The constant $-a$ in this solution is the solution of the equation $r + a = 0$. Since differentiating an exponential e^{rx} any number of times, where r is a constant,always yields a constant times e^{rx}, it is reasonable to expect that for some appropriate constant r, e^{rx} will be a solution of the equation (1.2). Let us try it for (1.2). We formulate the result as a theorem.

Theorem 1.1. *Let a_1, a_2 be constants, and consider the equation*

$$L(y) = y'' + a_1y' + a_2y = 0$$

If r_1, r_2 are distinct roots of the characteristic polynomial p, where

$$p(r) = r^2 + a_1r + a_2,$$

then the functions ϕ_1, ϕ_2 defined by

$$\phi_1 = e^{r_1x}, \qquad \phi_2 = e^{r_2x} \tag{1.3}$$

are solutions of $L(y) = 0$. If r_1 is a repeated root of the p, then the functions ϕ_1, ϕ_2 defined by

$$\phi_1 = e^{r_1x}, \qquad \phi_2 = xe^{r_1x} \tag{1.4}$$

are solutions of $L(y) = 0$.

Proof. Consider the equation $L(y) = y'' + a_1y' + a_2y = 0$ where a_1, a_2 are constants.

Now, consider the function e^{rx}. Then we find that

$$\begin{aligned} L(e^{rx}) &= (e^{rx})'' + a_1(e^{rx})' + a_2(e^{rx}) \\ &= r^2 e^{rx} + a_1 r e^{rx} + a_2 e^{rx} \\ &= (r^2 + a_1 r + a_2) e^{rx}, \end{aligned}$$

and e^{rx} will be a solution of $L(y) = 0$, i.e. $L(e^{rx}) = 0$, if it satisfies $r^2 + a_1 r + a_2 = 0$. We let

$$p(r) = r^2 + a_1 r + a_2,$$

and call p the *characteristic polynomial* of L, or of the equation (1.2). Note that $p(r)$ can be obtained from $L(y)$ by replacing y^k everywhere by r^k, where we use the conventions that the zero-th derivative of $y, y^{(0)}$, is y itself and that $r^0 = 1$. From the fundamental theorem of Algebra, we know that the polynomial p always has two complex roots r_1, r_2 (which may be real). If $r_1 \neq r_2$, then $p(r_1) = 0 = p(r_2)$. Therefore $L(e^{r_1 x}) = 0 = L(e^{r_2 x})$. Hence $e^{r_1 x}$ and $e^{r_2 x}$ are two distinct solutions of $L(y) = 0$. It is possible to find two distinct solutions in the case $r_1 = r_2$ also. Since r_1 is a root of p(r), $p(r_1) = 0$. Hence $e^{r_1 x}$ is one solution of $L(y) = 0$. Also, We have

$$L(e^{rx}) = p(r)e^{rx} \tag{1.5}$$

for all r and x. We recall that if r_1 is a repeated root of p, then not only $p(r_1) = 0$, but $p'(r_1) = 0$. This suggests differentiating the equation (1.3) with respect to r. Then we observe that since L involves only differentiation with respect to x,

$$\frac{\partial}{\partial r} L(e^{rx}) = L\left(\frac{\partial}{\partial r}(e^{rx})\right) = L(xe^{rx}),$$

and therefore

$$\begin{aligned} L(xe^{rx}) &= (xe^{rx})'' + a_1(xe^{rx})' + a_2(xe^{rx}) \\ &= xr^2 e^{rx} + 2re^{rx} + a_1 e^{rx} + a_1 xre^{rx} + a_2 xe^{rx} \\ &= \left[xr^2 + 2r + a_1 + a_1 xr + a_2 x\right] e^{rx} \\ &= \left[p'(r) + xp(r)\right] e^{rx} \end{aligned}$$

Now setting $r = r_1$ in this equation we see that $L(xe^{r_1 x}) = 0$, thus showing that $xe^{r_1 x}$ is another solution in case $r_1 = r_2$. Hence the theorem. □

Remark 1.2. If ϕ_1, ϕ_2 are any two solutions of $L(y) = 0$, c_1, c_2 are any two constants, then the linear combination of two solutions $\phi = c_1\ \phi_1 + c_2\ \phi_2$ is also a solution of the equation $L(y) = 0$. Indeed

$$\begin{aligned} L(\phi) &= (c_1\ \phi_1 + c_2\ \phi_2)'' + a_1(c_1\ \phi_1 + c_2\ \phi_2)' + a_2(c_1\ \phi_1 + c_2\ \phi_2) \\ &= c_1\phi_1'' + c_2\phi_2'' + c_1a_1\phi_1' + c_2a_2\phi_2' + c_1a_1\phi_1 + c_2a_2\phi_2 \\ &= c_1\ L(\phi_1) + c_2\ L(\phi_2) \\ &= 0 \end{aligned}$$

The function ϕ which is zero for all x is also a solution, the trivial solution of $L(y) = 0$.

Example 1.3. Consider the equation $\phi = y'' + y' - 2y = 0$.
The characteristic polynomial is $p(r) = r^2 + r - 2$ and its roots are -2 and 1.
Every solution ϕ is of the form $\phi(x) = c_1 e^{-2x} + c_2 e^x$ where c_1, c_2 are constants.

Example 1.4. Consider the equation $y'' + \omega^2 y = 0$ where ω is a positive constant.
The characteristic polynomial is $p(r) = r^2 + \omega^2$ and its roots are $i\omega$ and $-i\omega$.
Every solution ϕ is of the form $\phi(x) = c_1 e^{i\omega x} + c_2 e^{-i\omega x}$ where c_1, c_2 are constants.
Taking $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$ we see that

$$\begin{aligned} \phi(x) &= \frac{1}{2}e^{i\omega x} + \frac{1}{2}e^{-i\omega x} \\ &= \frac{e^{i\omega x} + e^{-i\omega x}}{2} \\ &= \cos\omega x \end{aligned}$$

Therefore $\cos\omega x$ is a solution. Similarly, taking $c_1 = \frac{1}{2i}$ and $c_2 = \frac{-1}{2i}$ we see that

$$\begin{aligned} \phi(x) &= \frac{1}{2i}e^{i\omega x} + \frac{-1}{2i}e^{-i\omega x} \\ &= \frac{e^{i\omega x} - e^{-i\omega x}}{2i} \\ &= \sin\omega x \end{aligned}$$

Therefore $\sin\omega x$ is a solution. The equation $y'' + \omega^2 y = 0$ is called the harmonic oscillator equation.

Exercise:

1. Find the solution of the following equations.

(i) $y'' - 4y = 0$ (ii) $3y'' + 2y' = 0$

(iii) $y'' = 0$ (iv) $y'' + 16y = 0$
(v) $y'' + 2iy' + y = 0$ (vi) $y'' - 4y' + 5y = 0$

2. Consider the equation $y'' + y' - 6y = 0$
   (a) Compute the solution ϕ satisfying $\phi(0) = 1,\ \phi'(0) = 0$.
   (b) Compute the solution ψ satisfying $\psi(0) = 1,\ \psi'(0) = 0$.
   (c) Compute $\phi(1)$ and $\psi(1)$.
3. Find all solutions ϕ of $y'' + y = 0$ satisfying
(i) $\phi(0) = 0,\ \phi(\pi) = 0$ (ii)$\phi(0) = 1,\ \phi'(\pi/2) = 2$
(iii) $\phi(0) = 0,\ \phi'(\pi/2) = 0$ (iv) $\phi(0) = 0,\ \phi(\pi/2) = 0$.

1.3 Initial value problems

Every solution of the equation

$$L(y) = y'' + a_1 y' + a_2 y = 0$$

is a linear combination of the solutions (1.4) or (1.5) will depend on showing that the initial value problems for this equation have unique solutions. An *initial value problem* for $L(y) = 0$ is a problem of finding a solution ϕ satisfying

$$\phi(x_0) = \alpha, \qquad \phi'(x_0) = \beta, \tag{1.6}$$

where x_0 is some real number, and $\alpha,\ \beta$ are two given constants. Thus we specify ϕ and its first derivative at some initial point x_0. This problem is denoted by

$$L(y) = 0, \qquad y(x_0) = \alpha, \qquad y'(x_0) = \beta, \tag{1.7}$$

Theorem 1.5. (Existence Theorem) *For any real x_0, and constants $\alpha,\ \beta$, there exists a solution ϕ of lhe initial value problcm (1.7) on $-\infty < x < \infty$.*

Proof. We show that there are unique constants c_1, c_2 such that $\phi = c_1\phi_1 + c_2\phi_2$ satisfies (1.6), where ϕ_1, ϕ_2 are the solutions given by (1.3) or (1.4). In order to satisfy the relations (1.6) we must have

$$\begin{aligned} \phi(x_0) &= c_1\phi_1(x_0) + c_2\phi_2(x_0) = \alpha \\ \phi'(x_0) &= c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = \beta \end{aligned} \tag{1.8}$$

By representing the equations (1.8) in the matrix form, we have

$$\begin{pmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

This matrix equation will have unique solution c_1, c_2 if the determinant

$$\Delta = \begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{vmatrix} = \phi_1(x_0)\phi_2'(x_0) - \phi_1'(x_0)\phi_2(x_0) \neq 0.$$

In case $r_1 \neq r_2$,

$$\phi_1 = e^{r_1 x}, \qquad \phi_2 = e^{r_2 x},$$

and

$$\Delta = r_2 e^{r_1 x_0} e^{r_2 x_0} - r_1 e^{r_1 x_0} e^{r_2 x_0} = (r_2 - r_1) e^{(r_1 + r_2) x_0},$$

which is not zero, since $e^{(r_1+r_2)x_0} \neq 0$ and $r_1 \neq r_2$.

In case $r_1 = r_2$,

$$\phi_1 = e^{r_1 x}, \qquad \phi_2 = x e^{r_1 x},$$

and

$$\Delta = e^{r_1 x_0}(e^{r_1 x_0} + x_0 r_1 e^{r_1 x_0}) - r_1 x_0 e^{r_1 x_0} e^{r_1 x_0} = e^{2 r_1 x_0} \neq 0.$$

Therefore the determinant condition is satisfied in either case. Thus, if c_1, c_2 are the unique constants satisfying (1.8), the function

$$\phi = c_1 \phi_1 + c_2 \phi_2$$

will be the desired solution satisfying $\phi(x_0) = \alpha, \qquad \phi'(x_0) = \beta.$ □

We have shown that there is a unique linear combination of ϕ_1 and ϕ_2 which is a solution of (1.7). Although it is not quite obvious, it turns out that this solution is the only one. Before proving this we give an estimate for the rate of growth of any solution ϕ of $L(y) = 0$, and its first derivative ϕ', in terms of the coefficients $1, a_1, a_2$ appearing in $L(y)$. As a measure of the "size" of ϕ and ϕ' we take

$$\|\phi(x)\| = (\ |\phi(x)|^2 + |\phi'(x)|^2)^{1/2},$$

where the positive square root is understood. The "size" of L will be measured by

$$k = 1 + |a_1| + |a_2|.$$

Note that If b and c are any two constants, the we have the inequality that

$$2\ |b|\ |c| \leq |b|^2 + |c|^2. \tag{1.9}$$

This inequality results by noticing that

$$0 \le (|b| - |c|)^2 = |b|^2 + |c|^2 - 2|b||c|.$$

Theorem 1.6. *Let ϕ be any solution of $L(y) = y'' + a_1 y' + a_2 y = 0$ on an interval I containing a point x_0. Then for all x in I*

$$\|\phi(x_0)\| \ e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| \ e^{k|x-x_0|} \tag{1.10}$$

where $\|\phi(x)\| = \left(\ |\phi(x)|^2 + |\phi'(x)|^2 \right)^{1/2}, \ \ k = 1 + |a_1| + |a_2|.$

Proof. **Remark:** Geometrically the inequality (1.10) says that $\|\phi(x)\|$ always remains between the two curves.

$$y = \|\phi(x_0)\| \ e^{-k(x-x_0)} \text{ and } y = \|\phi(x_0)\| \ e^{k(x-x_0)};$$

We let $u(x) = \|\phi(x)\|^2$ for $x \in I$. Then

$$\begin{aligned} u(x) &= |\phi(x)|^2 + |\phi'(x)|^2 \\ &= \phi(x)\overline{\phi(x)} + \phi'(x)\overline{\phi'(x)}, \text{ since } |z|^2 = z\overline{z} \\ &= \phi(x)\overline{\phi}(x) + \phi'(x)\overline{\phi'}(x) \end{aligned}$$

Then

$$\begin{aligned} u'(x) &= \phi(x)\overline{\phi'}(x) + \phi'(x)\overline{\phi}(x) + \phi'(x)\overline{\phi''}(x) + \phi''(x)\overline{\phi'}(x) \\ |u'(x)| &= |\phi(x)\overline{\phi'}(x) + \phi'(x)\overline{\phi}(x) + \phi'(x)\overline{\phi''}(x) + \phi''(x)\overline{\phi'}(x)| \\ &\le |\phi(x)\overline{\phi'}(x)| + |\phi'(x)\overline{\phi}(x)| + |\phi'(x)\overline{\phi''}(x)| + |\phi''(x)\overline{\phi'}(x)| \\ &= |\phi(x)||\overline{\phi'}(x)| + |\phi'(x)||\overline{\phi}(x)| + |\phi'(x)||\overline{\phi''}(x)| + |\phi''(x)||\overline{\phi'}(x)| \\ &= |\phi(x)||\phi'(x)| + |\phi'(x)||\phi(x)| + |\phi'(x)||\phi''(x)| + |\phi''(x)||\phi'(x)| \\ &= 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||\phi''(x)| \\ &\le 2\ |\phi(x)||\phi'(x)| + 2\ |\phi'(x)|\ (|a_1||\phi'(x)| + |a_2||\phi(x)|) \\ &= 2\ (1 + |a_2|)\ |\phi(x)||\phi'(x)| + 2\ |a_1||\phi'(x)|^2 \\ &\le (1 + |a_2|)\ (|\phi(x)|^2 + |\phi'(x)|^2) + 2\ |a_1||\phi'(x)|^2, \ \ \text{using}(1.9) \\ &= (1 + |a_2|)\ |\phi(x)|^2 + (1 + |a_2|)\ |\phi'(x)|^2 + 2\ |a_1||\phi'(x)|^2 \\ &= (1 + |a_2|)\ |\phi(x)|^2 + (1 + |a_2| + 2\ |a_1|)\ |\phi'(x)|^2 \\ &\le 2\ (1 + |a_1| + |a_2|)\ |\phi(x)|^2 + (1 + |a_1| + |a_2|)\ |\phi'(x)|^2 \\ &= 2\ (1 + |a_1| + |a_2|)\ (|\phi(x)|^2 + |\phi'(x)|^2) \\ &= 2\ k\ u(x), \ \ \text{where } k = 1 + |a_1| + |a_2| \end{aligned}$$

Therefore $|u'(x)| \leq 2\ k\ u(x)$. That is $-2\ k\ u(x) \leq u'(x) \leq 2\ k\ u(x)$.
Take $u'(x) \leq 2\ k\ u(x)$. Then

$$\begin{aligned} u'(x) - 2\ k\ u(x) &\leq 0 \\ e^{-2kx}u'(x) + u(x)(-2ke^{-2kx}) &\leq 0 \\ (e^{-2kx}\ u(x))' &\leq 0 \end{aligned}$$

Let $x_0 < x$

$$\begin{aligned} \int_{x_0}^{x} (e^{-2kt}\ u(t))'\ dt &\leq 0 \\ e^{-2kx}\ u(x) - e^{-2kx_0}\ u(x_0) &\leq 0 \\ e^{-2kx}\ u(x) &\leq e^{-2kx_0}\ u(x_0) \\ u(x) &\leq e^{2k(x-x_0)}\ u(x_0) \\ \|\phi(x)\|^2 &\leq \|\phi(x_0)\|^2\ e^{2k(x-x_0)} \\ \|\phi(x)\| &\leq \|\phi(x_0)\|\ e^{k(x-x_0)} \end{aligned}$$

Similarly taking $-2\ k\ u(x) \leq u'(x)$ we can show that $\|\phi(x_0)\|\ e^{-k(x-x_0)} \leq \|\phi(x)\|$. Hence

$$\|\phi(x_0)\|\ e^{-k(x-x_0)} \leq \|\phi(x)\| \leq \|\phi(x_0)\|\ e^{k(x-x_0)} \quad \text{when } x_0 < x \tag{1.11}$$

In a similar way, we can show that

$$\|\phi(x_0)\|\ e^{-k(x_0-x)} \leq \|\phi(x)\| \leq \|\phi(x_0)\|\ e^{k(x_0-x)} \quad \text{when } x < x_0 \tag{1.12}$$

Hence from (1.11) and (1.12), we have

$$\|\phi(x_0)\|\ e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\|\ e^{k|x-x_0|}$$

where $k = 1 + |a_1| + |a_2|$. Hence the theorem. □

Theorem 1.7. (Uniqueness Theorem) Let α, β be any tw0 constants, and let x_0 be any real number. On any interval I containing x_0 there exists at most one solution ϕ of the initial value problem $L(y) = 0,\ y(x_0) = \alpha,\ y'(x_0) = \beta$.

Proof. Suppose ϕ and ψ are two solutions of the initial value problem $L(y) = y'' + a_1y' + a_2y = 0,\ y(x_0) = \alpha,\ y'(x_0) = \beta$. Then we have to prove that $\phi(x) = \psi(x)$

for all x. Let $\chi = \phi(x) - \psi(x)$. Then $L(\chi) = L(\phi) - L(\psi) = 0$, and $\chi(x_0) = 0$, $\chi'(x_0) = 0$. Then $\|\chi(x_0)\| = 0$, and applying the inequality (1.10) to χ we see that $\|\chi(x)\| = 0$ for all x in I. This implies that $\chi(x) = 0$ for all x in I, or $\phi = \psi$, proving our result. □

Theorem 1.8. *Let ϕ, ψ be the two solutions of $L(y) = 0$ given by (1.3) in case $r_1 \neq r_2$, and by (1.4) in case $r_1 = r_2$. If c_1, c_2 are any two constants the function $\phi = c_1\phi_1 + c_2\phi_2$ is a solution of $L(y) = 0$ on $-\infty < x < \infty$. Conversely, if ϕ is any solution of $L(y) = 0$ on $-\infty < x < \infty$, there are unique constants c_1, c_2 such that $\phi = c_1\phi_1 + c_2\phi_2$.*

Example 1.9. Find the solution of the initial value problem
$y'' - 2y' - 3y = 0,\ y(0) = 0\ y'(0) = 1.$
Soln:
The characteristic polynomial is $r^2 - 2r - 3$ and its roos are $3, -1$. Then
$y(x) = c_1e^{3x} + c_2e^{-x}$ is a solution of given equation.

Also $y(0) = 0$ implies that $c_1 + c_2 = 0$ and $y'(0) = 1$ implies that $3c_1 - c_2 = 1$. solving these we have $c_1 = \frac{1}{4}$ and $c_2 = \frac{-1}{4}$.

Hence $y(x) = \frac{1}{4}e^{3x} - \frac{1}{4}e^{-x}$ is a solution of the given initial value problem.

Exercise:
1. Find the solution of the following initial value problem.
(i) $y'' + 10y = 0,\ y(0) = \pi,\ y'(0) = \pi^2$
(ii) $y'' + (3i - 1)y' - 3iy = 0,\ y(0) = 2,\ y'(0) = 0$

1.4 Linear dependence and independence

Two functions ϕ_1, ϕ_2 defined on an interval I are said to be *linearly dependent* on I if there exist two constants c_1, c_2, not both zero, such that $c_1\phi_1(x) + c_2\phi_2(x) = 0$ for all x in I. The functions ϕ_1, ϕ_2, are said to be *linearly independent* on I if they are not linearly dependent there. Thus ϕ_1, ϕ_2 are lincarly independent on I if the only constants c_1, c_2 Such that $c_1\phi_1(x) + c_2\phi_2(x) = 0$ for all x in I are the constants $c_1 = 0, c_2 = 0$.

The functions defined by (1.3) are linearly independent on any interval I. For suppose

$$c_1e^{r_1}x + c_2e^{r_2}x = 0 \tag{1.13}$$

for all x in I. Then, multiplying by e^{-r_1x}, we obtain

$$c_1 + c_2e^{(r_2-r_1)x} = 0,$$

and differentiating these results

$$c_2(r_2 - r_1)e^{(r_2-r_1)x} = 0.$$

Since $r_1 \neq r_2$, and $e^{(r_2-r_1)x}$ is never zero, this implies $c_2 = 0$. But if $c_2 = 0$, the relation (1.13) gives $c_1e^{r_1x} = 0$, or $c_1 = 0$ also.

Similarly the functions ϕ_1, ϕ_2 defined by (1.4) are linearly independent on any interval I. The proof is the same. If

$$c_1e^{r_1x} + c_2xe^{r_1}x = 0$$

on I, by multiplying by e^{-r_1x} we get $c_1 + c_2x = 0$, and differentiating we obtain $c_2 = 0$, and this implies $c_1 = 0$.

There is a simple test which enables us to tell whether two solutions ϕ_1, ϕ_2, of $L(y) = 0$ are linearly independent or not. It involves the determinant

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2$$

which is called the *Wronskian* of ϕ_1, ϕ_2. It is a function, and its value at x is denoted by $W(\phi_1, \phi_2)(x)$.

Theorem 1.10. *Two solutions ϕ_1, ϕ_2 of $L(y) = 0$ are linearly independent on an interval I if and only if $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I.*

Proof. First suppose $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I, and let c_1, c_2 be constants such that

$$c_1\phi_1(x) + c_2\phi_2(x) = 0 \tag{1.14}$$

for all x in I. Then also

$$c_1\phi_1'(x) + c_2\phi_2'(x) = 0 \tag{1.15}$$

for all x in I. For a fixed x the equations (1.14), (1.15) are linear homogeneous equations satisfied by c_1, c_2. Hence the matrix representation of the equations (1.14) and (1.15) is

$$\begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the determinant of the coefficients of c_1, c_2 in (1.14) and (1.15) is just $W(\phi_1, \phi_2)(x)$ which is not zero. Therefore the matrix $\begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{pmatrix}$ is non-singular. Hence the above matrix equation has unique solution namely $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. That is $c_1 = 0,\ c_2 = 0$. Therefore $c_1 = 0,\ c_2 = 0$ is the only solution of (1.14) and (1.15). This proves that ϕ_1, ϕ_2 are linearly independent on I.

Conversely, assume ϕ_1, ϕ_2 are linearly independent on I. Suppose that there is an x_0 in I such that $W(\phi_1, \phi_2)(x_0) = 0$. This implies that the system of two equations

$$\begin{aligned} c_1\phi_1(x_0) + c_2\phi_2(x_0) &= 0 \\ c_1\phi_1'(x_0) + c_2\phi_2'(x_0) &= 0 \end{aligned} \tag{1.16}$$

has a solution c_1, c_2, where at least one of these numbers is not zero. Let c_1, c_2 be such a solution and consider the function $\psi = c_1\phi_1 + c_2\phi_2$. Now $L(\psi) = 0$, and from (1.16) we see that

$$\psi(x_0) = 0,\ \ \psi'(x_0) = 0.$$

From the Uniqueness theorem (Theorem 1.7), we infer that $\psi(x) = 0$ for all x in I and thus

$$c_1\phi(x) + c_2\phi_2(x) = 0$$

for all x in I. But this contradicts the fact that ϕ_1, ϕ_2 are linearly independent on I. Thus the superposition that there was a point x_0 in I such that $W(\phi_1, \phi_2) = 0$ must be false. We have consequently proved that $W(\phi_1, \phi_2) \neq 0$ for all x in I. □

It is easy to see that we need compute $W(\phi_1, \phi_2)$ at only one convenient point to test the linear independence of the solutions ϕ_1, ϕ_2.

Theorem 1.11. *Let ϕ_1, ϕ_2 be two solutions of $L(y) = 0$ on an interval I and let x_0 be any point in I. Then ϕ_1, ϕ_2 are linearly independent on I if and only if $W(\phi_1, \phi_2)(x_0) \neq 0$.*

Proof. If ϕ_1, ϕ_2 are linearly independent on I then $W(\phi_1, \phi_2) \neq 0$ for all x in I by Theorem 1.10. In particular, $W(\phi_1, \phi_2)(x_0) \neq 0$

Conversely, suppose $W(\phi_1, \phi_2)(x_0) \neq 0$, and suppose c_1, c_2 are constants such that

$$c_1\phi_1 + c_2\phi_2 = 0$$

for all x in I. Then we see that

$$\begin{aligned} c_1\phi_1(x_0) + c_2\phi_2(x_0) &= 0 \\ c_1\phi_1'(x_0) + c_2\phi_2'(x_0) &= 0 \end{aligned} \tag{1.17}$$

and since the determinant of the coefficients is $W(\phi_1, \phi_2) \neq 0$, we obtain $c_1 = c_2 = 0$. Thus ϕ_1, ϕ_2 are linearly independent on I. □

Using the concept of linear independence we can show any two linearly independent solutions of $L(y) = 0$ determine all solutions, in the sense of the following theorem.

Theorem 1.12. *Let ϕ_1, ϕ_2 be any two linearly independent solutions of $L(y) = 0$ on an interval I. Every solulion ϕ of $L(y) = 0$ can be written uniquely as $\phi = c_1\phi_1 + c_2\phi_2$, where c_1, c_2 are constants.*

Remark 1.13. The importance of Theorem 1.12 is that we need only to find any two linearly independent solutions of $L(y) = 0$ in order to obtain all solutions of $L(y) = 0$.

Example 1.14. Consider the equation $y'' + y = 0$. Its characteristic polynomial is $r^2 + 1$ and its roots are $i, -i$. Hence it has two solutions e^i and e^{-i}, which are linearly independent, since the wronskian of two functions id non-zero for all x. But it also has the two linearly independent solutions $\cos x, \sin x$. Sometimes it is more convenient to express a solution in terms of the latter set of functions, especially when we want to observe the oscillatory character of a real-valued solution.

Exercise:

The functions ϕ_1, ϕ_2, defined below exist for $-\infty < x < \infty$. Determine whether they are linearly dependent or independent there.

(i) $\phi_1(x) = x^2$, $\phi_2(x) = 5x^2$

(ii) $\phi_1(x) = \sin x$, $\phi_2(x) = e^{ix}$

(iii) $\phi_1(x) = \cos x$, $\phi_2(x) = 3(e^{ix} + e^{-ix})$

1.5 A Formula for the Wronskian

There is a convenient formula for the Wronskian of two solutions of $L(y) = 0$, which results from the fact that $W(\phi_1, \phi_2)$ satisfies a first order linear equation.

Theorem 1.15. *If ϕ_1, ϕ_2 are two solutions of $L(y) = 0$ on an interval I containing a point x_0, then*

$$W(\phi_1, \phi_2)(x) = e^{-a_1(x-x_0)}\, W(\phi_1, \phi_2)(x_0). \tag{1.18}$$

Proof. Let ϕ_1, ϕ_2 be two solutions of $L(y) = 0$. Then we have

$$\phi_1'' + a_1\phi_1' + a_2\phi_1 = 0 \text{ and } \phi_2'' + a_1\phi_2' + a_2\phi_2 = 0$$

and mulplying the first equation by $-\phi_2$, the second by ϕ_1 and adding we obtain

$$(\phi_1\phi_2'' - \phi_1''\phi_2) + a_1(\phi_1\phi_2' - \phi_1'\phi_2) = 0.$$

we notice that if $W = W(\phi_1, \phi_2)$,

$$W = \phi_1\phi_2' - \phi_1'\phi_2, \text{ and } W' = \phi_1\phi_2'' - \phi_1''\phi_2.$$

Thus W satisfies the first order equation

$$W' + a_1 W = 0.$$

Hence $W(x) = ce^{-a_1 x}$, where c is some constant. Setting $x = x_0$ we see that

$$W(x_0) = ce^{-a_1 x_0},$$

or

$$c = e^{-a_1 x_0} W(x_0),$$

and thus

$$W(x) = e^{-a_1(x-x_0)} W(x_0),$$

which was to be proved. □

1.6 Non-homogeneous equation of order two

We turn now to the problem of finding all solutions of the equation

$$L(y) = y'' + a_1 y' + a_2 y = b(x),$$

where b is some continuous function on an interval I. Suppose we know that ψ_p is a particular solution of this equation, and that ψ is any other solution. Then

$$L(\psi - \psi_p) = L(\psi) - L(\psi_p) = b - b = 0$$

on I. This shows that $\psi - \psi_p$ is a solution of the homogeneous equation $L(y) = 0$. Therefore if ϕ_1, ϕ_2 are linearly independent solutions of $L(y) = 0$, there are unique costants c_1, c_2 Such that

$$\psi - \psi_p = c_1\phi_1 + c_2\phi_2.$$

In other words every solution ψ of $L(y) = b(x)$ can be written in the form

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2$$

and we see that the problem of finding all solutions of $L(y) = b(x)$ reduces to finding a particular one ψ_p, and two linearly independent solutions ϕ_1, ϕ_2 of $L(y) = 0$. Note that if

$$L(\psi_p) = b \text{ and } L(\phi_1) = L(\phi_2) = 0.$$

and c_1, c_2 are any constants, then

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2$$

satisfies $L(\psi) = b$.

To find a particular solution of $L(y) = b(x)$ we reason in the following way. Every solution of $L(y) = 0$ is of the form $c_1\phi_1 + c_2\phi_2$ where c_1, c_2 are constants, and ϕ_1, ϕ_2 are linearly independent solutions. Such a function $c_1\phi_1 + c_2\phi_2$ can not be a solution of $L(y) = b(x)$ unless $b(x) = 0$ on I. However, suppose we allow c_1, c_2 to become functions u_1, u_2 (not necessarily constants) on I, and then ask whether there is a solution of $L(y) = b(x)$ of the form $u_1\phi_1 + u_2\phi_2$ on I. This procedure is known as the *variation of constants*. The remarkable thing is that it works. We argue in reverse. Suppose we have a solution of $L(y) = b(x)$ of the form $u_1\phi_1 + u_2\phi_2$, where u_1, u_2 are functions. Then

$$\begin{aligned}
&(u_1\phi_1 + u_2\phi_2)'' + a_1(u_1\phi_1 + u_2\phi_2)' + a_2(u_1\phi_1 + u_2\phi_2) \\
&= u_1L(\phi_1) + u_2L(\phi_2) + (\phi_1u_1'' + \phi_2u_2'') + 2(\phi_1'u_1' + \phi_2'u_2') + a_2(\phi_1u_1' + \phi_2u_2') \\
&= +(\phi_1u_1'' + \phi_2u_2'') + 2(\phi_1'u_1' + \phi_2'u_2') + a_2(\phi_1u_1' + \phi_2u_2') = b
\end{aligned}$$

and we noticed that if

$$\phi_1u_1' + \phi_2u_2' = 0 \qquad (1.19)$$

then

$0 = (\phi_1u_1' + \phi_2u_2')' = (\phi_1'u_1' + \phi_2'u_2') + (\phi_1u_1'' + \phi_2u_2'')$, and we must have

$$\phi_1'u_1' + \phi_2'u_2' = b. \qquad (1.20)$$

looking at this reasoning in reverse we see that if we can find two functions u_1, u_2

satisfying (1.19), (1.20), then indeed $u_1\phi_1 + u_2\phi_2$ will satisfy $L(y) = b(x)$.

The equations (1.19), (1.20) are two linear equations for u_1', u_2', with a determinant which is just the Wronskian $W(\phi_1, \phi_2)$. Since we assumed ϕ_1, ϕ_2 to be linearly independent this determinant is never zero on I, and there exist unique solutions u_1', u_2'. Indeed, a little calculation shows that

$$u_1' = \frac{\phi_2\ b}{W(\phi_1, \phi_2)}, \qquad u_2' = \frac{\phi_1\ b}{W(\phi_1, \phi_2)}.$$

In order to obtain u_1, u_2 all we have to do is integrate. For example, if x_0 is in I we may take for u_1, u_2

$$u_1(x) = -\int_{x_0}^{x} \frac{\phi_2(t)\ b(t)}{W(\phi_1, \phi_2)(t)}\ dt, \qquad u_2(x) = \int_{x_0}^{x} \frac{\phi_1(t)\ b(t)}{W(\phi_1, \phi_2)(t)}\ dt.$$

The solution $\psi_p = u_1\phi_1 + u_2\phi_2$ then takes the form

$$\psi_p(x) = \int_{x_0}^{x} \frac{[\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)]\ \ b(t)}{W(\phi_1, \phi_2)(t)}\ dt. \tag{1.21}$$

We summarize our results,

Theorem 1.16. *Let b be continuous on an interval I. Every solution ψ of $L(y) = b(x)$ on I can be written as*

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2$$

where ψ_p is a particular solution, ϕ_1, ϕ_2 are two linearly independent solutions of $L(y) = 0$, and c_1, c_2 are constants. A particular solution ψ_p, is qiven by (1.21). Conversely every such ψ is a solution of $L(y) = b(x)$.

Example 1.17. Consider the equation $y'' - y' - 2y = e^{-x}$.

The characteristic polynomial is

$$r^2 - r - 2 = (r + 1)(r - 2),$$

and therefore two linearly independent solutions ϕ_1, ϕ_2 of the homogeneous equation are

$$\phi_1(x) = e^{-x}, \qquad \phi_2(x) = e^{2x}.$$

A particular solution ψ_p of the non-homogeneous equation is of the form

$$\psi_p(x) = u_1(x)e^{-x} + u_2(x)e^{2x},$$

where u_1', u_2' satisfies the equations (1.19) and (1.20)

$$\phi_1 u_1' + \phi_2 u_2' = 0 \text{ and } \phi_1' u_1' + \phi_2' u_2' = b.$$

That is,

$$u_1'e^{-x} + u_2'e^{2x} = 0$$
$$-u_1'e^{-x} + 2u_2'e^{2x} = e^{-x}.$$

The matrix representation of above equations are

$$\begin{pmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-x} \end{pmatrix}$$

Then the wronskian is

$$W(\phi_1, \phi_2)(x) = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2\ e^{2x} \end{vmatrix} = 3\ e^{x}$$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{3\ e^{x}} \begin{pmatrix} 2e^{2x} & -e^{2x} \\ e^{-x} & e^{-x} \end{pmatrix} \begin{pmatrix} 0 \\ e^{-x} \end{pmatrix} = \frac{1}{3\ e^{x}} \begin{pmatrix} -e^{x} \\ e^{-2x} \end{pmatrix}$$

Therefore $u_1' = \dfrac{1}{3\ e^{x}}\ (-e^{x})$ and $u_2' = \dfrac{1}{3\ e^{x}}\ (e^{-2x})$. Then on integration we obtain

$$u_1 = \frac{-1}{3}x \text{ and } u_2 = \frac{-1}{9}e^{-3x}.$$

Hence the particular integral

$$\psi_p = u_1\phi_1 + u_2\phi_2 = -\frac{x}{3}\ e^{-x} - \frac{1}{9}e^{-x}.$$

Thus the general solution ψ of the non-homogeneous equation has the form

$$\psi = c_1e^{-x} + c_2e^{2x} - \frac{x}{3}\ e^{-x} - \frac{1}{9}e^{-x}.$$

where c_1, c_2 are any two constants.

Exercise: Solve the following equations

(a) $4y'' - y = e^{x}$ (b) $y'' - 7y' + 6y = \sin x$

(c) $y'' + 4y = \cos x$ (d) $y'' - 4y' + 5y = 3e^{-x} + 2x^2$

(e) $y'' + 9y = \sin 3x$ (f) $6y'' + 5y' - 6y = x$

Chapter 2

Linear equations with constant coefficients

2.1 The homogeneous equation of order n

Everything we have done for the second order equation can be carried over to the case of the equation of order n. Now let $L(y)$ be given by

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y,$$

where $a_1, a_2, \cdots, a_n$ are constants. We try to solve $L(y) = 0$ as before by trying an exponential e^{rx}. We see that

$$L(e^{rx}) = p(r)e^{rx}, \tag{2.1}$$

where $p(r) = r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n$.

We call p the *characteristic polynomial* of L. If r_1 is a root ot p, then clearly $L(e^{r_1 x}) = 0$, and we have a solution $e^{r_1 x}$. If r_1 is a root of multiplicity m_1 of p, then

$$p(r_1) = 0, \quad p'(r_1) = 0, \quad p''(r_1) = 0, \cdots, p^{(m_1-1)}(r_1) = 0.$$

If we differentiate the equation (2.1) k times with respect to r, we obtain

$$\begin{aligned}
\frac{\partial^k}{\partial r^k} L(e^{rx}) &= L\left(\frac{\partial^k}{\partial r^k}(e^{rx})\right) = L(x^k e^{rx}) \\
&= \left[p^{(k)}(r) + kp^{(k-1)}(r)x + \frac{k(k-1)}{2\,!}p^{(k-2)}(r)x^2 + \cdots + p(r)x^k\right] e^{rx}.
\end{aligned}$$

Note: If f and g are two functions having k derivatives, then

$$(fg)^{(k)} = f^{(k)}g + kf^{(k-1)}g' + \frac{k(k-1)}{2\,!}f^{(k-2)}g'' + \cdots + fg^{(k)}.$$

Thus for $k = 0, 1, \cdots, m_1 - 1$, we see that $x^k e^{r_1 x}$ is a solution of $L(y) = 0$. Repeating this proress for each root of p we arrive at the following result.

Theorem 2.1. *Let $r_1, r_2, \cdots, r_s$, be the distinct roots of the characteristic polynomial p, and suppose r_i has multiplicity m_i (thus $m_1 + m_2 + \cdots + m_s = n$). Then the n functions*

$$\begin{gathered} e^{r_1 x}, xe^{r_1 x}, \cdots, x^{m_1-1}e^{r_1 x}; \\ e^{r_2 x}, xe^{r_2 x}, \cdots, x^{m_2-1}e^{r_2 x}; \cdots; \\ e^{r_s x}, xe^{r_s x}, \cdots, x^{m_s-1}e^{r_s x} \end{gathered}$$

are solutions of $L(y) = 0$.

Definition 2.2. The n functions $\phi_1,\ \phi_2,\ \cdots,\ \phi_n$ on an interval I are said to be *linearly dependent* on I if there are constants $c_1, c_2, \cdots, c_n$ not all zero, such that

$$c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n = 0$$

for all x in I. The functions $\phi_1, \phi_2, \cdots, \phi_n$ are said to be *linearly independent* on I if they are not linearly dependent on I.

Theorem 2.3. *The n solutions of $L(y) = 0$ given by*

$$\begin{gathered} e^{r_1 x}, xe^{r_1 x}, \cdots, x^{m_1-1}e^{r_1 x}; \\ e^{r_2 x}, xe^{r_2 x}, \cdots, x^{m_2-1}e^{r_2 x}; \cdots; \\ e^{r_s x}, xe^{r_s x}, \cdots, x^{m_s-1}e^{r_s x} \end{gathered}$$

are linearly independent on any interval I.

Proof. Suppose we have n constants

$$c_{ij} \quad (i = 1, 2, \cdots, s; \quad j = 0, 1, \cdots, m_i - 1)$$

such that

$$\sum_{i=1}^{s}\sum_{j=0}^{m_i-1} c_{ij}\ x^j\ e^{r_i x} \tag{2.2}$$

on I. Summing over j for fixed i, we let

$$P_i(x) = \sum_{j=0}^{m_i-1} c_{ij}\ x^j$$

be the polynomial coefficient of $e^{r_i x}$ in (2.2). Thus we have

$$P_1(x)e^{r_1 x} + P_2(x)e^{r_2 x} + \cdots + P_s(x)e^{r_s x} = 0 \tag{2.3}$$

on I. Assume that not all the constants c_{ij} are 0. Then there will be at least one of the polynomials P_i which is not identically zero on I. By relabeling the roots r_i if necessary we can assume that P_s, is not identically zero on I. Now (2.3) implies that

$$P_1(x) + P_2(x)e^{(r_2-r_1)x} + \cdots + P_s(x)e^{(r_s-r_1)x} = 0 \tag{2.4}$$

on I. Upon differentiating (2.4) sufficiently many times (at most m_1 times) we can reduce $P_1(x)$ to 0. In this procoss the degrees of the polynomials multiplying $e^{(r_s-r_1)x}$ remain unchanged, as well as the non-identically vanishing character of any of these polynomials. We obtain an expression of the form

$$Q_2(x)e^{(r_2-r_1)x} + Q_3(x)e^{(r_3-r_1)x} + \cdots + Q_s(x)e^{(r_s-r_1)x} = 0$$

or

$$Q_2(x)e^{r_2x} + Q_3(x)e^{r_3x} + \cdots + Q_s(x)e^{r_sx} = 0$$

on I, where the Q_i are polynomials, $\deg Q_i = \deg P_i$, and Q_s does not vanish identically. Continuing this process we finally arrive at a situation where

$$R_s(x)e^{r_sx} = 0 \tag{2.5}$$

on I, and R_s is a polynomial, $\deg R_s = \deg P_s$, which does not vanish identically on I. But (2.5) implies that $R_s(x) = 0$ for all x on I. This contradiction forces us to abandon the supposition that P_s is not identically zero. Thus $P_s(x) = 0$ for all x in 1, and we have shown that all the constants $c_{ij} = 0$, proving that the n solutions given in Theorem are linearly independent on any interval I. □

Example 2.4. Consider the equation $y''' - 3y' + 2y = 0$.

The characteristic polynomial is $p(r) = r^3 - 3r + 2$ and its roots are $1, 1, -2$. Thus three linearly independent solutions are given by

$$e^x, \qquad x\ e^x, \qquad e^{-2x},$$

and any solution ϕ has the form

$$\phi(x) = (c_1 + c_2x)e^x + c_3e^{-2x},$$

where c_1, c_2, c_3 are any constants.

Exercise: Determine whether the set of functions defined on $-\infty < x < \infty$ are linearly independent or dependent.
(a) $\phi_1(x) = 1, \quad \phi_2(x) = x \quad \phi_3(x) = x^2$
(b) $\phi_1(x) = e^{ix}, \quad \phi_2(x) = \sin x \quad \phi_3(x) = 2\cos x$

(c) $\phi_1(x) = x, \quad \phi_2(x) = e^{2x} \quad \phi_3(x) = |x|$.

Exercise: Find the solutions of the following equations:

(a) $y''' - 8y = 0$ (b) $y^{(4)} + 16y = 0$

(c) $y''' - 5y'' + 6y' = 0$ (d) $y^{(4)} + 5y'' + 4y = 0$

(e) $y''' - 3y' - 2y = 0$ (f) $y^(4) - 16y = 0$

Exercise: Compute the wronskian of four linearly independent solutions of the equation $y^{(4)} + 16y = 0$.

2.2 Initial value problems for n-th order equations

An initial value problem for $L(y) = 0$ is a problem of finding a solution which has prescribed values for it, and its first $n-1$ derivatives, at some point x_0 (the initial point). If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are given constants, and x_0 is some real number, the problem of finding a solution ϕ of $L(y) = 0$ satisfying

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n,$$

is denoted by

$$L(y) = 0, y(x_0) = \alpha_1, y'(x_0) = \alpha_2, \cdots, y^{(n-1)}(x_0) = \alpha_n.$$

There is only one solution to such an initial value problem, and the demonstration of this will depend on an estimate for the rate of growth of a solution ϕ of $L(y) = 0$, together with its derivatives $\phi', \cdots, \phi^{(n-1)}$. We define $\|\phi(x)\|$ by

$$\phi(x) = \left[|\phi(x)|^2 + \cdots + \left|\phi^{(n-1)}(x)\right|^2\right]^{1/2},$$

positive square root being understood, and give the analogue of Theorem 1.6.

Theorem 2.5. *Let ϕ be any solution of $L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$ on an interval I containing a point x_0. Then for all x in I,*

$$\|\phi(x_0)\| \; e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| \; e^{k|x-x_0|} \tag{2.6}$$

where $\|\phi(x)\| = \left(\, |\phi(x)|^2 + |\phi'(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2\right)^{1/2}$,
$k = 1 + |a_1| + |a_2| + \cdots + |a_n|$.

Proof. Letting $u(x) = \|\phi(x)\|^2$ for $x \in I$. Then

$$\begin{aligned}
u(x) &= |\phi(x)|^2 + |\phi'(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2 \\
&= \phi(x)\overline{\phi(x)} + \phi'(x)\overline{\phi'(x)} + \cdots + \phi^{(n-1)}(x)\overline{\phi^{(n-1)}(x)}, \text{ since } |z|^2 = z\overline{z} \\
&= \phi(x)\overline{\phi}(x) + \phi'(x)\overline{\phi'}(x) + \cdots + \phi^{(n-1)}(x)\overline{\phi^{(n-1)}}(x)
\end{aligned}$$

Then

$$\begin{aligned}
u'(x) &= \phi(x)\overline{\phi'}(x) + \phi'(x)\overline{\phi}(x) + \phi'(x)\overline{\phi''}(x) + \phi''(x)\overline{\phi'}(x) \\
&\quad + \cdots + \phi^{(n)}(x)\overline{\phi^{(n-1)}}(x) + \phi^{(n-1)}(x)\overline{\phi^{(n)}}(x) \\
|u'(x)| &= |\phi(x)\overline{\phi'}(x) + \phi'(x)\overline{\phi}(x) + \phi'(x)\overline{\phi''}(x) + \phi''(x)\overline{\phi'}(x) \\
&\quad + \cdots + \phi^{(n)}(x)\overline{\phi^{(n-1)}}(x) + \phi^{(n-1)}(x)\overline{\phi^{(n)}}(x)| \\
&\leq |\phi(x)\overline{\phi'}(x)| + |\phi'(x)\overline{\phi}(x)| + |\phi'(x)\overline{\phi''}(x)| + |\phi''(x)\overline{\phi'}(x)| \\
&\quad + \cdots + |\phi^{(n)}(x)\overline{\phi^{(n-1)}}(x)| + |\phi^{(n-1)}(x)\overline{\phi^{(n)}}(x)| \\
&= |\phi(x)||\overline{\phi'}(x)| + |\phi'(x)||\overline{\phi}(x)| + |\phi'(x)||\overline{\phi''}(x)| + |\phi''(x)||\overline{\phi'}(x)| \\
&\quad + \cdots + |\phi^{(n)}(x)||\overline{\phi^{(n-1)}}(x)| + |\phi^{(n-1)}(x)||\overline{\phi^{(n)}}(x)| \\
&= |\phi(x)||\phi'(x)| + |\phi'(x)||\phi(x)| + |\phi'(x)||\phi''(x)| + |\phi''(x)||\phi'(x)| \\
&\quad + \cdots + |\phi^{(n)}(x)||\phi^{(n-1)}(x)| + |\phi^{(n-1)}(x)||\phi^{(n)}(x)| \\
&= 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||\phi''(x)| + \cdots + 2|\phi^{(n-1)}(x)||\phi^{(n)}(x)| \\
&\leq 2\ |\phi(x)||\phi'(x)| + 2\ |\phi'(x)||\phi''(x)| + \cdots + \\
&\quad 2|\phi^{(n-1)}(x)|\ \left(|a_1||\phi^{(n-1)}(x)| + |a_2||\phi^{(n-2)}(x)| + \cdots + |a_n||\phi(x)|\right) \\
&\leq (1 + |a_n|)|\phi(x)|^2 + (2 + |a^{(n-1)}|)|\phi'(x)|^2 + \cdots + \\
&\quad (2 + |a_1|)|\phi^{(n-2)}(x)|^2 + (1 + 2|a_1| + |a_2| + \cdots + |a_n|)|\phi^{(n-1)}(x)|^2, \quad \text{using(1.9)} \\
&\leq 2(1 + |a_1| + |a_2| + \cdots + |a_n|)|\phi(x)|^2 + 2(1 + |a_1| + |a_2| + \cdots + |a_n|)|\phi'(x)|^2 \\
&\quad + \cdots + 2(1 + |a_1| + |a_2| + \cdots + |a_n|)|\phi^{(n-2)}(x)|^2 \\
&\quad + 2(1 + |a_1| + |a_2| + \cdots + |a_n|)|\phi^{(n-1)}(x)|^2 \\
&= 2(1 + |a_1| + |a_2| + \cdots + |a_n|)\ \left(|\phi(x)|^2 + |\phi'(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2\right) \\
&= 2\ k\ u(x), \quad \text{where } k = 1 + |a_1| + |a_2| + \cdots + |a_n|
\end{aligned}$$

Therefore $|u'(x)| \leq 2\ k\ u(x)$. That is $-2\ k\ u(x) \leq u'(x) \leq 2\ k\ u(x)$.

Take $u'(x) \leq 2\ k\ u(x)$. Then

$$\begin{aligned} u'(x) - 2\ k\ u(x) &\leq 0 \\ e^{-2kx}u'(x) + u(x)(-2ke^{-2kx}) &\leq 0 \\ (e^{-2kx}\ u(x))' &\leq 0 \end{aligned}$$

Let $x_0 < x$

$$\begin{aligned} \int_{x_0}^{x} (e^{-2kt}\ u(t))'\ dt &\leq 0 \\ e^{-2kx}\ u(x) - e^{-2kx_0}\ u(x_0) &\leq 0 \\ e^{-2kx}\ u(x) &\leq e^{-2kx_0}\ u(x_0) \\ u(x) &\leq e^{2k(x-x_0)}\ u(x_0) \\ \|\phi(x)\|^2 &\leq \|\phi(x_0)\|^2\ e^{2k(x-x_0)} \\ \|\phi(x)\| &\leq \|\phi(x_0)\|\ e^{k(x-x_0)} \end{aligned}$$

Similarly taking $-2\ k\ u(x) \leq u'(x)$ we can show that $\|\phi(x_0)\|\ e^{-k(x-x_0)} \leq \|\phi(x)\|$. Hence

$$\|\phi(x_0)\|\ e^{-k(x-x_0)} \leq \|\phi(x)\| \leq \|\phi(x_0)\|\ e^{k(x-x_0)} \quad \text{when } x_0 < x \tag{2.7}$$

In a similar way, we can show that

$$\|\phi(x_0)\|\ e^{-k(x_0-x)} \leq \|\phi(x)\| \leq \|\phi(x_0)\|\ e^{k(x_0-x)} \quad \text{when } x < x_0 \tag{2.8}$$

Hence from (2.7) and (2.8), we have

$$\|\phi(x_0)\|\ e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\|\ e^{k|x-x_0|}$$

where $k = 1 + |a_1| + |a_2| + \cdots + |a_n|$. Hence the theorem. □

Theorem 2.6. (Uniqueness Theorem) Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be any n constants, and let x_0 be any real number. On any interval I containing x_0 there exists at most one solution ϕ of $L(y) = 0$ satisfying

$$\phi(x_0) = \alpha_1, \quad \phi'(x_0) = \alpha_2, \quad \cdots, \quad \phi^{(n-1)}(x_0) = \alpha_n.$$

Proof. The proof is the same as that of Theorem 1.7.
Suppose ϕ and ψ are two solutions of the initial value problem $L(y) = 0,\ y(x_0) = \alpha_1,\ y'(x_0) = \alpha_2, \quad \cdots, \quad y^{(n-1)}(x_0) = \alpha_n$ on I. Then we have to prove that $\phi(x) =$

$\psi(x)$ for all x. Let $\chi = \phi(x) - \psi(x)$. Then $L(\chi) = L(\phi) - L(\psi) = 0$, and $\chi(x_0) = 0,\ \chi'(x_0) = 0,\ \cdots,\ \chi^{(n-1)}(x_0) = 0$. Then $\|\chi(x_0)\| = 0$, and applying the inequality (2.6) to χ we see that $\|\chi(x)\| = 0$ for all x in I. This implies that $\chi(x) = 0$ for all x in I, or $\phi = \psi$, proving our result. □

Definition 2.7. The wronskian $W(\phi_1, \phi_2, \cdots, \phi_n)$ of n functions $\phi_1, \phi_2, \cdots, \phi_n$ having $n-1$ derivatives on an interval I is defined to be the determinant function

$$W(\phi_1, \phi_2, \cdots, \phi_n) = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_n' \\ \vdots & & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}$$

Theorem 2.8. *If $\phi_1, \phi_2, \cdots, \phi_n$ are solutions of $L(y) = 0$ on an interval I, then they are linearly independent there if and only if $W(\phi, \phi_2, \cdots, \phi_n)(x) \neq 0$ for all x in I.*

Proof. First suppose $W(\phi_1, \phi_2, \cdots, \phi_n)(x) \neq 0$ for all x in I, and let $c_1, c_2, \cdots, c_n$ be constants such that

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0 \tag{2.9}$$

for all x in I. Then also

$$\begin{aligned} c_1\phi_1'(x) + c_2\phi_2'(x) + \cdots + c_n\phi_n'(x) &= 0 \\ c_1\phi_1''(x) + c_2\phi_2''(x) + \cdots + c_n\phi_n''(x) &= 0 \\ &\vdots \\ c_1\phi_1^{(n-1)}(x) + c_2\phi_2^{(n-1)}(x) + \cdots + c_n\phi_n^{(n-1)}(x) &= 0 \end{aligned} \tag{2.10}$$

for all x in I. For a fixed x the equations (2.9), (2.10) are linear homogeneous equations satisfied by $c_1, c_2, \cdots, c_n$. Hence the matrix representation of the equations (2.9) and (2.10) is

$$\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since the determinant of the coefficients of $c_1, c_2 \cdots, c_n$ in (2.9) and (2.10) is just $W(\phi_1, \phi_2, \cdots, \phi_n)(x)$ which is not zero. Therefore the matrix

$$\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix}$$

is non-singular. Hence the above matrix equation has unique solution namely $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. That is $c_1 = 0,\ c_2 = 0, \cdots, c_n = 0$. Therefore $c_1 = 0,\ c_2 = 0, \cdots, c_n = 0$ is the only solution of (2.9) and (2.10). This proves that $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I.

Conversely, assume ϕ_1, ϕ_2 are linearly independent on I. Suppose that there is an x_0 in I such that $W(\phi_1, \phi_2, \cdots, \phi_n)(x_0) = 0$. This implies that the system of linear equations

$$\begin{aligned} c_1\phi_1(x_0) + c_2\phi_2(x_0) + \cdots + c_n\phi_n(x_0) &= 0 \\ c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \cdots + c_n\phi_n'(x_0) &= 0 \\ c_1\phi_1''(x_0) + c_2\phi_2''(x_0) + \cdots + c_n\phi_n''(x_0) &= 0 \\ &\vdots \\ c_1\phi_1^{(n-1)}(x_0) + c_2\phi_2^{(n-1)}(x_0) + \cdots + c_n\phi_n^{(n-1)}(x_0) &= 0 \end{aligned} \tag{2.11}$$

has a solution $c_1, c_2, \cdots, c_n$, where at least one of these numbers is not zero. Let $c_1, c_2, \cdots, c_n$ be such a solution and consider the function $\psi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$. Now $L(\psi) = 0$, and from (2.11) we see that

$$\psi(x_0) = 0,\ \psi'(x_0) = 0, \cdots, \psi^{(n-1)}(x_0) = 0.$$

From the Uniqueness theorem (Theorem 2.6), we infer that $\psi(x) = 0$ for all x in I and thus

$$c_1\phi(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0$$

for all x in I. But this contradicts the fact that $\phi_1, \phi_2 \cdots, \phi_n$ are linearly independent on I. Thus the superposition that there was a point x_0 in I such that $W(\phi_1, \phi_2, \cdots, \phi_n) = 0$ must be false. We have consequently proved that $W(\phi_1, \phi_2, \cdots, \phi_n) \neq 0$ for all x in I. $\square$

Note: The above result and the proof do not depend on the fact that L has *constant coefficients.*

Theorem 2.9. (Existence Theorem) Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be any n constants, and let x_0 be any real number. There exists a solution ϕ of $L(y) = 0$ on $-\infty < x < \infty$ satisfying

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n \tag{2.12}$$

Proof. Let $\phi_1, \phi_2, \cdots, \phi_n$ be any set of n linearly independent solutions of $L(y) = 0$. We know that $\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$ os a solution of $L(y) = 0$.

$$\begin{aligned}
\phi(x_0) &= c_1\phi_1(x_0) + c_2\phi_2(x_0) + \cdots + c_n\phi_n(x_0) = \alpha_1 \\
\phi'(x_0) &= c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \cdots + c_n\phi_n'(x_0) = \alpha_2 \\
\phi''(x_0) &= c_1\phi_1''(x_0) + c_2\phi_2''(x_0) + \cdots + c_n\phi_n''(x_0) = \alpha_3 \\
&\vdots \\
\phi^{(n-1)}(x_0) &= c_1\phi_1^{(n-1)}(x_0) + c_2\phi_2^{(n-1)}(x_0) + \cdots + c_n\phi_n^{(n-1)}(x_0) = \alpha_n
\end{aligned} \tag{2.13}$$

Therefore the matrix representation is

$$\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Since $W(\phi_1, \phi_2, \cdots, \phi_n)(x) \neq 0$, the matrix

$$\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix}$$

is non-singular. Hence the above matrix equation has unique solution. Therefore there exists unique set of constants $c_1, c_2, \cdots, c_n$ satisfying (2.13). For this choice of $c_1, c_2, \cdots, c_n$ the function $\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$ will be the desired solution. $\square$

Theorem 2.10. *Let $\phi_1, \phi_2, \cdots, \phi_n$ be n linearly independent solutions of $L(y) = 0$ on an interval I. If $c_1, c_2, \cdots, c_n$ are any constants*

$$\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n \tag{2.14}$$

is a solution, and every solution may be represented in this form.

Proof. We have already seen that

$$L(\phi) = c_1 L(\phi_1) + \cdots + c_n L(\phi_n) = 0.$$

Now, let ϕ be any solution of $L(y) = 0$, and let x_0 be in I. Suppose

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n.$$

In the proof of Theorem 2.9 we showed that there exist unique constants $c_1, c_2, \cdots, c_n$ Such that $\psi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$ is a solution of $L(y) = 0$ on I satisfying

$$\psi(x_0) = \alpha_1, \psi'(x_0) = \alpha_2, \cdots, \psi^{(n-1)}(x_0) = \alpha_n.$$

The uniqueness theorem (Theorem 2.6) implies that $\phi = \psi$, proving that ϕ may be represented as in (2.14) □

A simple formula exists for the Wronskian, as in the case $n = 2$.

Theorem 2.11. *If $\phi_1, \phi_2, \cdots, \phi_n$ are two solutions of $L(y) = 0$ on an interval I containing a point x_0, then*

$$W(\phi_1, \phi_2, \cdots, \phi_n)(x) = e^{-a_1(x-x_0)}\, W(\phi_1, \phi_2, \cdots, \phi_n)(x_0). \tag{2.15}$$

Proof. Let $\phi_1, \phi_2, \cdots, \phi_n$ be n solutions of $L(y) = 0$.

Let $W = W(\phi_1, \phi_2, \cdots, \phi_n)$,

From the definition of W, its derivatives W' is the sum of n determinants. That is, $W' = V_1 + V_2 + \cdots + V_n$, where V_k differ from W only in its k^{th}throw and k^{th} row of V_k is obtained by differentiating k^{th} row of W. Thus

$$W' = \begin{vmatrix} \phi_1' & \phi_2' & \cdots & \phi_n' \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1'' & \phi_2'' & \cdots & \phi_n'' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} + \cdots +$$

$$\begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-2)} & \phi_2^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ \phi_1^{(n)} & \phi_2^{(n)} & \cdots & \phi_n^{(n)} \end{vmatrix}$$

The first $n-1$ determinants $V_1, V_2, \cdots, V_{n-1}$ are all zero, since they each have two identical rows. Since $\phi_1, \phi_2, \cdots, \phi_n$ are solutions of $L(y) = 0$, we have

$$\phi_i^{(n)} = -a_1\phi_i^{(n-1)} - a_2\phi_i^{(n-2)} - \cdots - a_n\phi_1.$$

Therefore,

$$W' = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ -\sum_{j=0}^{n-1} a_{n-j}\phi_1^{(j)} & -\sum_{j=0}^{n-1} a_{n-j}\phi_2^{(j)} & \cdots & -\sum_{j=0}^{n-1} a_{n-j}\phi_n^{(j)} \end{vmatrix}$$

The value of this determinant is unchanged if we multiply any row by a number and add to the last row. Hence

$$W' = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ -a_1\phi_1^{(n-1)} & -a_1\phi_2^{(n-1)} & \cdots & -a_1\phi_n^{(n-1)} \end{vmatrix}$$

$$= -a_1 \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} = -a_1\ W$$

Thus W satisfies the first order equation $W' + a_1W = 0$.

Hence $W(x) = ce^{-a_1x}$, where c is some constant. Setting $x = x_0$ we see that

$$W(x_0) = ce^{-a_1x_0},$$

or

$$c = e^{a_1x_0}W(x_0),$$

and thus

$$W(\phi_1, \phi_2, \cdots, \phi_n)(x) = e^{-a_1(x-x_0)} W(\phi_1, \phi_2, \cdots, \phi_n)(x_0),$$

which was to be proved. □

Corollary 2.12. *Let $\phi_1, \phi_2, \cdots, \phi_n$ be n solutions of $L(y) = 0$ on an interval I containing x_0. Then they are linearly independent on I if and only if* $W(\phi_1, \phi_2, \cdots, \phi_n)(x_0) \neq 0$.

Proof. The proof is an immediate consequence of Theorem 2.8 and the formula (2.15). □

Example 2.13. As an illustration of the use of wronskian formula (2.15), we consider the homogeneous equation of order 3 which has a root r_1 with multiplicity 3. Its characteristic polynomial is

$$p(r) = (r - r_1)^3 = r^3 - 3r_1 r^2 + 3r_1^2 r - r_1^2.$$

Hence $L(y) = y''' - 3r_1 y'' + 3r_1^2 y' - r_1^2 y$

and we have $a_1 = -3r_1$. We take

$$\phi_1(x) = e^{r_1 x},\ \phi_2(x) = x\ e^{r_1 x},\ \phi_3(x) = x^2\ e^{r_1 x}$$

and then obtain

$$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} e^{r_1 x} & x\ e^{r_1 x} & x^2\ e^{r_1 x} \\ r_1 e^{r_1 x} & (1 + r_1 x)e^{r_1 x} & (2x + r_1 x^2)e^{r_1 x} \\ r_1^2 e^{r_1 x} & (2r_1 + r_1^2 x)e^{r_1 x} & (2 + 4r_1 x + r_1^2 x^2)e^{r_1 x} \end{vmatrix}$$

This a little involved to evaluate directly, but using (2.15) with $x_0 = 0$ we obtain

$$W(\phi_1, \phi_2, \phi_3)(0) = \begin{vmatrix} 1 & 0 & 0 \\ r_1 & 1 & 0 \\ r_1^2 & 2r_1 & 2 \end{vmatrix} = 2$$

and hence $W(\phi_1, \phi_2, \phi_3)(x) = 2e^{3r_1 x}$.

2.3 Equations with real constants

Suppose that the constants $a_1, a_2, \cdots, a_n$ in

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$$

are all real numbers. The characteristic polynomial

$$p(r) = r^n + a_1 r^{n-1} + \cdots + a_n$$

then has all real coeficients. This implies that

$$\overline{p(r)} = p(\overline{r}) \tag{2.16}$$

for all r, since

$$\begin{aligned} \overline{p(r)} &= \overline{r^n + a_1 r^{n-1} + \cdots + a_n} \\ &= \overline{r^n} + \overline{a_1 r^{n-1}} + \cdots + \overline{a_n} \\ &= \overline{r}^{(n)} + \overline{a}_1 \overline{r}^{n-1} + \cdots + \overline{a}_n \\ &= \overline{r}^n + a_1 \overline{r}^{n-1} + \cdots + a_n \\ &= p(\overline{r}) \end{aligned}$$

Fron (2.16) it follows that if r_1 is a root of p, then so is $\overline{r}_1$. Thus the roots of p whose imaginary parts do not vanish occur in conjugate pairs. A slight extension of this argument shows that if r_1 is a root of multiplicity m_1, then $\overline{r}_1$ is a root with the same multiplicity m_1. If there are s distinct roots of p, let us enumerate them as follows:

$$r_1, \ \overline{r}_1, \ r_2, \ \overline{r}_2, \ \cdots, \ r_j, \ \overline{r}_j, \ r_{2j+1}, \ \cdots, \ r_s$$

where

$$r_k = \alpha_k + i\ \beta_k, \qquad (k = 1, 2, \cdots, j\ ;\ \alpha_k,\ \beta_k - \text{ real}\ ;\ r_k \neq 0)$$

and $r_{2j+1}, \cdots, r_s$ are real. Suppose that r_k has multiplicity m_k. Then we have

$$2(m_1 + m_2 + \cdots + m_j) + m_{2j+1} + \cdots + m_s = n.$$

Corresponding to these roots we have the n linearly independent solutions

$$e^{r_1 x}, xe^{r_1 x}, \cdots, x^{m_1 - 1} e^{r_1 x}\ ;\ e^{r_2 x}, xe^{r_2 x}, \cdots, x^{m_2 - 1} e^{r_2 x}\ ;\ \cdots\ ;\ e^{r_s x}, xe^{r_s x}, \cdots, x^{m_s - 1} e^{r_s x} \tag{2.17}$$

of $L(y) = 0$. Every solution is a linear combination, with constant coefficients, of these. We now note that if $1 \leq k \leq j,\ 0 \leq h \leq m_k - 1$,

$$\begin{aligned} x^k e^{r_k x} &= x^k e^{(\alpha_k + i\beta_k)x} = x^k e^{\alpha_k x}(\cos\beta_k x + i\sin\beta_k x), \\ x^k e^{\overline{r}_k x} &= x^k e^{(\alpha_k - i\beta_k)x} = x^k e^{\alpha_k x}(\cos\beta_k x - i\sin\beta_k x). \end{aligned} \tag{2.18}$$

Thus every solution is a linear combination, with constant coefficients, of the n

functions

$$
\begin{aligned}
e^{\alpha_1 x}\ \cos\beta_1 x,\ xe^{\alpha_1 x}\ \cos\beta_1 x,\ \cdots,\ x^{m_1-1}e^{\alpha_1 x}\ \cos\beta_1 x;\\
e^{\alpha_1 x}\ \sin\beta_1 x,\ xe^{\alpha_1 x}\ \sin\beta_1 x,\ \cdots,\ x^{m_1-1}e^{\alpha_1 x}\ \sin\beta_1 x;\\
\vdots\\
e^{r_s x},\ xe^{r_s x},\ \cdots,\ x^{m_s-1}e^{r_s x}.
\end{aligned}
\tag{2.19}
$$

Each of the functions in (2.19) is a solution of $L(y) = 0$ since, from (2.18),

$$
\begin{aligned}
x^h e^{\alpha_k x}\ \cos\beta_k x &= \frac{1}{2}x^h(e^{r_k x} + e^{\bar{r}_k x}),\\
x^h e^{\alpha_k x}\ \sin\beta_k x &= \frac{1}{2i}x^h(e^{r_k x} - e^{\bar{r}_k x}).
\end{aligned}
\tag{2.20}
$$

The solutions in (2.19) are all *real-valued*, and they are linearly independent. For suppose we have a linear combination of these functions equal to zero. Let us denote the terms in this sum which involve

$$x^h e^{\alpha_k x}\ \cos\beta_k x, \quad x^h e^{\alpha_k x}\ \sin\beta_k x$$

by

$$cx^h e^{\alpha_k x}\ \cos\beta_k x + dx^h e^{\alpha_k x}\ \sin\beta_k x,$$

where c and d are constants. Using (2.20) we find that we have a linear combination of the functions (2.17) equal to zero, and the terms involving $x^h e^{r_k x},\ x^h e^{\bar{r}_k x}$ will be

$$\frac{c - id}{2}x^h e^{r_k x} + \frac{c + id}{2}x^h e^{\bar{r}_k x}.$$

Since the functions (2.17) are linearly independent we must have all the coefficients in this sum equal to zero. In particular

$$c + id = 0, \qquad c - id = 0,$$

from which it follows that $c = 0,\ d = 0$. Thus the solutions (2.19) are linearly independent.

If ϕ is any *real-valued solution* of $L(y) = 0$, then ϕ is a linear combination of the real solutions (2.19) with *real* coefficients. Indeed, if we denote the solutions in (2.19) by $\phi_1, \cdots, \phi_n$, we have

$$\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n,$$

for some constants $c_1, c_2, \cdots, c_n$. Since $\phi_1, \phi_2, \cdots, \phi_n$ are all real-valued, we have

$$0 = \text{Im } \phi = (\text{Im } c_1)\phi_1 + (\text{Im } c_2)\phi_2 + \cdots + (\text{Im } c_n)\phi_n,$$

Nd since $\phi_1, \cdots, \phi_n$ are linearly independent we must have

$$\text{Im } c_1 = \text{Im } c_2 = \cdots = \text{Im } c_n = 0.$$

This shows that $c_1, c_2, \cdots, c_n$ are all real numbers.

Remark 2.14. If ϕ is a solution of $L(y) = 0$ which is such that

$$\phi(x_0) = \alpha_1, \quad \phi'(x_0) = \alpha_2, \quad \cdots, \quad \phi^{(n-1)}(x_0) = \alpha_n, \tag{2.21}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are real constants, then ϕ is real-valued. One way to see this is to note that since

$$L(\overline{\phi}) = \overline{L(\phi)} = 0,$$

$\overline{\phi}$ is also a solution, and hence so is

$$\psi = (1/2i)(\phi - \overline{\phi}) = \text{Im } \phi.$$

But, from (2.21) we see that

$$\psi(x_0) = 0, \; \psi'(x_0) = 0, \; , \cdots, \; \psi^{(n-1)}(x_0) = 0.$$

The uniqueness theorem implies that $\psi(x) = 0$ for all x, or Im $\phi = 0$, showing that ϕ is real-valued.

Theorem 2.15. *Suppose the constants $a_1, a_2, \cdots, a_n$ in the equation*

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

are all real. There exists a set of n linearly independent real-valued solutions (2.19), and every real-ralued solution is a linear combination of these with real coefficients. If a solution satisfies real initial conditions, it is real-valued.

Example 2.16. Consider the equation $y^{(4)} + y = 0$.

The characteristic polynomial is given by $p(r) = r^4 + 1$ and its roots are

$$\frac{1}{\sqrt{2}}(1+i), \; \frac{1}{\sqrt{2}}(1-i), \; \frac{1}{\sqrt{2}}(-1+i), \; \frac{1}{\sqrt{2}}(-1-i).$$

Thus every real solution of the given equation has the form

$$\begin{aligned} \phi(x) &= e^{x/\sqrt{2}} \left[c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2}) \right] \\ &+ e^{-x/\sqrt{2}} \left[c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2}) \right], \end{aligned}$$

where c_1, c_2, c_3, c_4 are real constants.

Exercise:

1. Find all real-valued solutions of the following equations:

(a) $y'' + y = 0$ (b) $y'' - y = 0$

(c) $y^{(4)} - y = 0$ (d) $y^{(5)} + 2y = 0$

(e) $y^{(4)} - 5y'' + 4y = 0$

2. Find the solution ϕ of the initial value problem
$y''' + y = 0,\ y(0) = 0\ y'(0) = 1,\ y''(0) = 0.$

3.Determine all real valued solutions of the equations:

(a) $y''' - iy'' + y' - iy = 0$

(b) $y'' - 2iy' - y = 0$

2.4 The non-homogeneous equation of order n

Let b be a continuous function on an interval I, and consider the equation
$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x),$$

where $a_1, a_2, \cdots, a_n$ are constants. If ψ_p is a particular solution of $L(y) = b(x)$ and ψ is any other solution, then
$$L(\psi - \psi_p) = L(\psi) - L(\psi_p) = b - b = 0.$$

Thus $\psi - \psi_p$ is a solution of the homogeneous equation $L(y) = 0$, and this inplies that any solution ψ of $L(y) = b(x)$ can be written in the form
$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

where ψ_p is a particular solution of $L(y) = b(x)$, the functions $\phi_1, \phi_2, \cdots, \phi_n$, are n linearly independent solutions of $L(y) = 0$, and $c_1, c_2, \cdots, c_n$ are constants.

To find a particular solution ψ_p, we proceed just as in the case $n = 2$, that is, we use the *variation of constants* method. We try to find n functions $u_1, u_2, \cdots u_n$ so that
$$\psi_p = u_1\phi_1 + u_2\phi_2 + \cdots + u_n\phi_n$$

is a solution. If
$$u_1'\phi_1 + u_2'\phi_2 + \cdots + u_n'\phi_n = 0,$$

then
$$\psi_p' = u_1\phi_1' + u_2\phi_2' \cdots + u_n\phi_n'$$

and if

$$u_1'\phi_1' + u_2'\phi_2' + \cdots + u_n'\phi_n' = 0,$$

then

$$\psi_p'' = u_1\phi_1'' + u_2\phi_2'' \cdots + u_n\phi_n''$$

Thus if $u_1', u_2', \cdots, u_n'$ satisfy

$$\begin{aligned} u_1'\phi_1 + u_2'\phi_2 + \cdots + u_n'\phi_n &= 0 \\ u_1'\phi_1' + u_2'\phi_2' + \cdots + u_n'\phi_n' &= 0 \\ &\vdots \\ u_1'\phi_1^{(n-2)} + u_2'\phi_2^{(n-2)} + \cdots + u_n'\phi_n^{(n-2)} &= 0 \\ u_1'\phi_1^{(n-1)} + u_2'\phi_2^{(n-1)} + \cdots + u_n'\phi_n^{(n-1)} &= b \end{aligned} \tag{2.22}$$

we see that

$$\begin{aligned} \psi_p &= u_1\phi_1 + u_2\phi_2 + \cdots + u_n\phi_n \\ \psi_p' &= u_1\phi_1' + u_2\phi_2' + \cdots + u_n\phi_n' \\ &\vdots \\ \psi_p^{(n-1)} &= u_1\phi_1^{(n-1)} + u_2\phi_2^{(n-1)} + \cdots + u_n\phi_n^{(n-1)} \\ \psi_p^{(n)} &= u_1\phi_1^{(n)} + u_2\phi_2^{(n)} + \cdots + u_n\phi_n^{(n)} + b \end{aligned} \tag{2.23}$$

Hence $L(\psi_p) = u_1L(\phi_1) + u_2L(\phi_2) + \cdots + u_nL(\phi_n) + b = b,$

and indeed ψ_p, is a solution of $L(y) = b(x)$. The whole problem is noW reduced to solving the linear system (2.22) for $u_1', u_2', \cdots, u_n'$. The determinant of the coefficients is just $W(\phi_1, \phi_2, \cdots, \phi_n)$, which is never zero when $\phi_1, \cdots, \phi_n$ ae linearly independent solutions of $L(y) = 0$. Therefore there are unique functions $u_1', \cdots, u_n'$, satisfying (2.22). It is easy to see that solutions are given by

$$u_k'(x) = \frac{W_k(x)\ b(x)}{W(\phi_1, \phi_2, \cdots, \phi_n)(x)}, \qquad (k = 1, 2, \cdots, n)$$

where W_k is the determinant obtained from $W(\phi_1, \cdots, \phi_n)$ by replacing the $k-$th column (that is $\phi_k, \phi_k', \cdots, \phi_k^{(n-1)}$) by $0, 0, \cdots, 0, 1$.

If x_0 is any point in I we may take for u_k the function given by

$$u_k(x) = \int_{x_0}^{x} \frac{W_k(t)\ b(t)}{W(\phi_1, \phi_2, \cdots, \phi_n)(t)}\ dt, \qquad (k = 1, 2, \cdots, n)$$

The particular solution ψ_p now takes the form

$$\psi_p = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{W_k(t)\ b(t)}{W(\phi_1, \phi_2, \cdots, \phi_n)(t)}\ dt. \tag{2.24}$$

Theorem 2.17. *Let b be continuous on an interval I, and let $\phi_1, \cdots, \phi_n$ be n linearly independent solutions of $L(y) = 0$ on I. Every solution ψ of $L(y) = b(x)$ can be written as*

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

where ψ_p is a particular solution of $L(y) = b(x)$ and $c_1, c_2, \cdots, c_n$ are constants. Every such ψ is a solution of $L(y) = b(x)$. A particular solution is given by (2.24).

Note: It is clear that the particular solution ψ_p given by (2.24) satisfies

$$\psi_p(x_0) = \psi_p'(x_0) = \cdots = \psi_p^{(n-1)}(x_0) = 0.$$

Example 2.18. Consider the equation

$$y''' + y'' + y' + y = 1 \tag{2.25}$$

which satisfies

$$psi_p(0) = 0, \quad \psi_p'(0) = 1, \quad \psi_p''(0) = 0. \tag{2.26}$$

The homogeneous equation is

$$y''' + y'' + y' + y = 0, \tag{2.27}$$

and the characteristic polynomial corresponding to it is

$$p(r) = r^3 + r^2 + r + 1.$$

The roots of p are $i, -i, -1$. Since we are interested in a solution satisfying real initial conditions we take for independent solutions of (2.27)

$$\phi_1(x) = \cos x, \quad \phi_2(x) = \sin x, \quad \phi_3(x) = e^{-x}.$$

To obtain a particular solution of (2.25) of the form $u_1\phi_1 + u_2\phi_2 + u_3\phi_3$, we must solve the following equations for u_1', u_2', u_3' :

$$\begin{aligned} u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 &= 0 \\ u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' &= 0 \\ u_1'\phi_1'' + u_2'\phi_2'' + u_3'\phi_3'' &= 1, \end{aligned}$$

vhich in this case reduce to

$$
\begin{aligned}
(\cos x)u_1' + (\sin x)u_2' + e^{-x}u_3' &= 0 \\
(-\sin x)u_1' + (\cos x)u_2' - e^{-x}u_3' &= 0 \\
(-\cos x)u_1' - (\sin x)u_2' + e^{-x}u_3' &= 1.
\end{aligned}
\tag{2.28}
$$

The determinant of the coefficients is

$$
W(\phi_1,\phi_2,\phi_3)(x) = \begin{vmatrix} \cos x & \sin x & e^{-x} \\ -\sin x & \cos x & -e^{-x} \\ -\cos x & -\sin x & e^{-x} \end{vmatrix}
$$

Using (2.15) we have

$$
W(\phi_1,\phi_2,\phi_3)(x) = e^{-x}\, W(\phi_1,\phi_2,\phi_3)(0),
$$

since $a_1 = 1$ in this case. Now

$$
W(\phi_1,\phi_2,\phi_3)(0) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = 2,
$$

and thus $W(\phi_1,\phi_2,\phi_3)(x) = 2e^{-x}$.

Solving (2.28) for u_1 we find that

$$
u_1'(x) = \frac{1}{2}\, e^{x} \begin{vmatrix} 0 & \sin x & e^{-x} \\ 0 & \cos x & -e^{-x} \\ 1 & -\sin x & e^{-x} \end{vmatrix} = -\frac{1}{2}\,(\cos x + \sin x). \tag{2.29}
$$

similarly we obtain

$$
u_2'(x) = \frac{1}{2}(\cos x - \sin x), \tag{2.30}
$$

$$
u_3'(x) = \frac{1}{2}\, e^{x}. \tag{2.31}
$$

Integrating (2.29),(2.30),(2.31), we obtain as choices u_1, u_2, u_3 :

$$
\begin{aligned}
u_1(x) &= \frac{1}{2}\,(\cos x - \sin x), \\
u_2(x) &= \frac{1}{2}(\sin x + \cos x), \\
u_3(x) &= \frac{1}{2}\, e^{x}.
\end{aligned}
$$

Therefore a particular solution of (2.25) is given by

$$\begin{aligned} u_1(x)\phi_1(x) \;&+\; u_2(x)\phi_2(x) \;+\; u_3(x)\phi_3(x) \\ &= \frac{1}{2}\,(\cos x - \sin x)\;\cos x + \frac{1}{2}(\sin x + \cos x)\;\sin x + \frac{1}{2}\,e^x \\ &= 1. \end{aligned}$$

The most general solution ψ of (2.25) is of the form

$$\psi(x) = 1 + c_1 \cos x + c_2 \sin x + c_3 e^{-x},$$

where c_1, c_2, c_3 are constants. We must choose these constants so that the conditions (2.26) are valid. This leads to the following equations for c_1, c_2, c_3 :

$$c_1 + c_2 = -1, \quad c_2 - c_3 = 1, \quad c_1 - c_3 = 0,$$

which have the unique solution

$$c_1 = -\frac{1}{2}, \quad c_2 = \frac{1}{2}, \quad c_3 = -\frac{1}{2}.$$

Therefore the solution of our problem is given by

$$\psi(x) = 1 + \frac{1}{2}(\sin x - \cos x - e^{-x}).$$

The solution corresponding to that given in (2.24), with $x_0 = 0$, is easily seen to be

$$\psi_p(x) = 1 - \frac{1}{2}(\cos x + \sin x + e^{-x}),$$

and this satisfies

$$\psi_p(0) = 0, \quad \psi_p'(0) = 0, \quad \psi_p''(0) = 0.$$

Exercise:

1. Find all solutions of the following equations:

(a) $y''' - y' = x$ (b) $y''' - 8y = e^{ix}$

(c) $y^{(4)} + 16y = \cos x$ (d) $y^{(4)} - y = \cos x$

2.5 A special method for solving the non-homogeneous equation

Although the variation of constants method yields a solution of the non-homogeneous equation it sometimes requires more labor than necessary. We now give a method, which often faster, of solving the non-homogeneous equation $L(y) = b(x)$ where b *is a solution of some homogeneous equation* $M(y) = 0$ *with conslant coefficients.* Thus

$b(x)$ must be a sum of terms of the type $P(x)\ e^{ax}$, where P is a polynomial and a is a constant.

Suppose L and M have constant coefficients, and have orders n and m respectively. If ψ is a solution of $L(y) = b(x)$, and $M(b) = 0$, then clearly

$$M(L(\psi)) = M(b) = 0.$$

This shows that ψ is a solution of a homogeneous equation $M(L(y)) = 0$ with constant coefficients of order $m + n$. Thus ψ can be written as a linear combination with constant coefficients of order $m+n$ linearly independent solutions of $M(L(y)) = 0$. Not every linear combination will be a solution of $L(y) = b(x)$ however. Thus, to find out what conditions must be satisfied by the constants, we substitute back into $L(y) = b(x)$. This always leads to a determination of a set of coefficients;

Example 2.19. Consider the equation

$$L(y) = y'' - 3y' + 2y = x^2.$$

Since x^2 is a solution of $M(y) = y''' = 0$, we see that every solution ψ of $L(y) = x^2$ is a solution of

$$M(L(y)) = y^{(5)} - 3y^{(4)} + 2y^{(3)} = 0.$$

The characteristic polynomial of this equation is $r^3(r^2 - 3r + 2)$ which is just the product of the characteristic polynomials for L and M. The roots are $0, 0, 0, 1, 2$ and hence ψ must have the form

$$\psi(x) = c_0 + c_1 x + c_2 x^2 + c_3 e^x + c_5 e^{2x}$$

We notice immediately that $c_4 e^x + c_5 e^{2x}$ is just a solution of $L(y) = 0$. Since we are interested only in a particular solution ψ_p of $L(y) = x^2$, we can assume ψ_p has the form

$$\psi_p = c_1 + c_2 x + c_3 x^2.$$

The problem is to determine the constants c_o, c_1, c_2 so that $L(\psi_p) = x^2$. Computing we find

$$\psi_p'(x) = c_1 + 2c_2 x, \quad \psi_p''(x) = 2c_2,$$

and

$$L(\psi_p) = (2c_2 - 3c_1 + 2c_0) + (-6c_2 + 2c_1)x + 2c_2 x^2 = x^2.$$

Thus equating coefficients of x^2, x and constants we have

$$2c_2 = 1, \quad -6c_2 + 2c_1 = 0 \quad \text{and} \quad 2c_2 - 3c_1 + 2c_0 = 0$$

Therefore $c_2 = \frac{1}{2}, \ \ c_1 = \frac{3}{2}, \ \ c_0 = \frac{7}{4}.$

Hence $\psi_p(x) = \frac{1}{4}(7 + 6x + 2x^2)$ is a particular solution of $L(y) = x^2$.

We call this method the *annihilator method*, since to solve $L(y) = b(x)$, we fnd an M which makes $M(b) = 0$, that is, *annihilates* b. Once M has been found the problem becomes algebraic in nature, no integrations being necessary. Actually, as we have seen from the cxample, all we require is the characteristic polynomial q of M. The following is a table of some functions together with charactecristic polynomials of annihilators. In this table a is onstant, and k is a non-ncgative integer.

	Function	Characteristic Polynomial of an Annihilator
(a)	e^{ax}	$r - a$
(b)	$x^k e^{ax}$	$(r - a)^{k+1}$
(c)	$\sin ax,\ \cos ax\ (a - \text{real})$	$r^2 + a^2$
(d)	$x^k \sin ax,\ x^k \cos ax\ (a - \text{real})$	$(r^2 + a^2)^{k+1}$

Let us consider another example of the annihilator method.

Example 2.20. Consider the equation

$$L(y) = y''' + y'' + y' + y = 1$$

Since 1 is a solution of $M(y) = y' = 0$, we see that every solution ψ of $L(y) = 1$ is a solution of

$$M(L(y)) = (y''' + y'' + y' + y)' = y^{(4)} + y''' + y'' + y' = 0.$$

The characteristic polynomial of this equation is

$$p(r) = r^4 + r^3 + r^2 + r = r(r^3 + r^2 + r + 1),$$

which is just the product of the characteristic polynomials for L and M. The roots are with roots $0, -1, i, -i$. and hence ψ must have the form

$$\psi(x) = c_0 + c_1 e^{-x} + c_2 \cos x + c_3 \sin x$$

We notice immediately that $c_1 e^{-x} + c_2 \cos x + c_3 \sin x$ is just a solution of $L(y) = 0$. Since we are interested only in a particular solution ψ_p of $L(y) = x^2$, we can assume ψ_p has the form

$$\psi_p = c_0.$$

The problem is to determine the constants c_0 so that $L(\psi_p) = 1$. Computing we find that $c_0 = 1$. For, since $L(\psi_p) = 1$, $\psi_p''' + \psi_p'' + \psi_p' + \psi_p = 1$.

Hence $\psi_p = 1$ is a particular solution of $L(y) = 1$.

Thus the general solution is $\psi(x) = 1 + c_1 e^{-x} + c_2 \cos x + c_3 \sin x$.

Exercise:

1. Using the annihilator method find a particular solution of each of the following equations:

(a) $y'' + 4y' = \cos x$ (b) $y'' + 4y = \sin 2x$

(c) $y'' + 9y = x^2 e^{3x}$ (d) $y'' + y = xe^x \cos 2x$

2.6 Algebra of constant coefficient operators

In order to justify the annihilator method we study the algebra of constant cocficient operators a little more carefully. For the type of equation we have in mind

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x),$$

where $a_0 \neq 0, a_1, \cdots, a_n$ are constants, and b is a sum of products of polynomials and exponentials, every solution ψ has all derivatives on $-\infty < x < \infty$. This follows from the fact that ψ has n derivatives there, and

$$\psi^{(n)} = b - \frac{a_1}{a_0}\psi^{(n-1)} - \cdots - \frac{a_n}{a_0}\psi,$$

where b has all derivatives on $-\infty < x < \infty$.

All the operators we now define will be assumed to be defined on the set of all functions ϕ on $-\infty < x < \infty$ which have all derivatives there. Let L and M denote the operators given by

$$L(\phi) = a_0\phi^{(n)} + a_1\phi^{(n-1)} + \cdots + a_n\phi,$$
$$M(\phi) = b_0\phi^{(m)} + b_1\phi^{(m-1)} + \cdots + b_m\phi,$$

where $a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_m$ are constants, with $a_0 \neq 0, b_0 \neq 0$. It will be convenient in what follows to consider a_0, b_0 which are not necessarily 1. The characteristic polynomials of L and M are thus

$$p(r) = a_0 r^n + a_1 r^{n-1} + a_n,$$

and

$$q(r) = b_0 r^m + b_1 r^{m-1} + \cdots + b_m,$$

respectively. We define the sum $L + M$ to be the operator given by

$$(L + M)(\phi) = L(\phi) + M(\phi),$$

and the product $M\ L$ to be the operator given by

$$(M\ L)(\phi) = M(L(\phi)).$$

If ϕ is a constant we define $\alpha\ L$ by

$$(\alpha\ L)(\phi) = \alpha(L(\phi)).$$

We note that $L + M$,ML and αL are all linear differential operators with constant coefficients.

Two operators L and M are said to be equal if

$$L(\phi) = M(\phi)$$

for all ϕ which have an infinite number of derivatives on $-\infty < x < \infty$. Suppose L, M have characteristic polynomials p, q respectively. Since e^{rx}, for any constant r, has an infinite number of derivatives on $-\infty < x < \infty$, we see that if $L = M$ then

$$L(e^{rx}) = p(r)e^{rx} = M(e^{rx}) = q(r)e^{rx},$$

and hence $p(r) = q(r)$ for all r. This implies that $m = n$, and $a_k = b_k, \quad k = 0, l, \cdots, n$. Thus $L = M$ if and only if L and M have the same order and the same coefficients, or, what is the same, if and only if $p = q$.

If D is the differentiation operator $D(\phi) = 0$, we define $D^2 = DD$, and successively $D^k = DD^{k-1}, \quad (k = 2, 3, \cdots)$.

For completeness we define D^0 by $D^0(\phi) = \phi$, but do not usually write it explicitly. If α is a constant we understand by α operating on a function ϕ just multiplication by α. Thus

$$\alpha(\phi) = (\alpha\ D^0)(\phi) = \alpha\phi.$$

Now, using our definitions, it is clear that

$$L = a_0 D^n + a_1 D^{n-1} + \cdots + a_n,$$

and

$$M = b_0 D^m + b_1 D^{m-1} + \cdots + b_m.$$

Theorem 2.21. *The correspondence which associates with each*

$$L = a_0 D^n + a_1 D^{n-1} + \cdots + a_n$$

its characteristic polynomial p given by

$$p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_n$$

is a one-to-one Correspondence between all linear differential operators with Constant coefficients and all polynomials. If L, M are associated with p, q respectively, then $L + M$ is associated with $p + q$, ML is associated with pq and αL is associated with αp (α a constant).

Proof. We have already seen that the correspondence is one-to-one since $L = M$ if and only if $p = q$. The remainder of the theorem can be shown directly or by noting that

$$(L + M)(e^{rx}) = L(e^{rx}) + M(e^{rx}) = [p(r) + q(r)]\,(e^{rx}),$$
$$(ML)(e^{rx}) = M(L(e^{rx})) = M(p(r)(e^{rx})) = p(r)M(e^{rx}) = p(r)q(r)(e^{rx}),$$
$$(\alpha L)(e^{rx}) = \alpha(L(e^{rx})) = \alpha p(r)(e^{rx})$$

This result implies that the algebraic properties of the constant coefficient operators are the same as those of the polynomials. For example, since LM and ML both have the characteristic polynomial pq. we have $LM = ML$. lf the roots of p are $r_1, r_2, \cdots, r_n$, then

$$p(r) = a_0(r - r_1)\cdots(r - r_n),$$

and since the operator $a_0(D - r_1)\cdots(D - r_n)$ has p as characteristic polynomial, we must have

$$L = a_0(D - r_1)\cdots(D - r_n).$$

This gives a factorization of L into a product of constant coefficient operators of the first order.

Remark 2.22. If L and M are not constant coefficient operators, then it may not be true that $LM = ML$. For example, if $L(\phi)(x) = \phi'(x),\quad M(\phi)(x) = x\phi(x)$, then $(LM - ML)(\phi)(x) = \phi(x)$.

We apply Theorem (2.21) to give a justification of the annilhilator method.

Theorem 2.23. *Consider the equation with constant coefficients*

$$L(y) = P(x)\ e^{ax}, \tag{2.32}$$

where P is the polynomial given by

$$P(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m, \quad (b_0 \neq 0) \tag{2.33}$$

Suppose a is a root of the characteristic polymomial p of L of multiplicity j. Then there is a unique solution ψ of (2.32) of lhe form

$$\psi(x) = x^j(c_0x^m + c_1x^{m-1} + \cdots + c_m)\ e^{ax},$$

where $c_0, c_1, \cdots, c_m$ are constants determined by the annihilator method.

Proof. The proof makes use of the formula

$$L(x^k e^{rx}) = \left[p(r)x^k + kp'(r)x^{k-1} + \frac{k(k-1)}{2\,!}\, p''(r)x^{k-2} + \cdots + kp^{(k-1)}(r)\ x + p^{(k)}(r)\right]\ e^{rx} \tag{2.34}$$

The coefficient of $P^{(l)}x^{k-l}$ in the bracket is the binomial coefficient

$$\binom{k}{l} = \frac{k\,!}{(k-l)\,!\,l\,!}.$$

Thus we may write

$$L(x^k e^{rx}) = \left[\sum_{l=0}^{k} \binom{k}{l}\ p^l(r)\ x^{k-l}\right]\ e^{rx}$$

where we understand $0! = 1$.

An annihilator of the right side of (2.32) is $M = (D-a)^{m+1}$, with characteristic polynomial given by $q(r) = (r-a)^{m+1}$.

Since a is a root of p with multiplicity j, it is a root of pq with multiplicity $j+m+1$. Thus solutions of $ML(y) = 0$ are of the form

$$\psi(x) = (c_0x^{j+m} + c_1x^{j+m-1} + \cdots + c_{j+m})\ e^{ax} + \phi(x),$$

where $L(\phi) = 0$, and ϕ involves exponentials of the form e^{sx}, with s a root of p and $s \neq a$. Since a is a root of p with multiplicity j, we have that

$$(c_{m+1}x^{j-1} + c_{m+2}x^{j-2} + \cdots + c_{m+j})\ e^{ax}$$

is also a solution of $L(y) = 0$. Consequently we see that there is a solution ψ of (2.32) having the form

$$\psi(x) = x^j(c_0x^m + c_1x^{m-1} + \cdots + c_m)e^{ax} \tag{2.35}$$

where $c_0, c_1, \cdots, c_m$ are constants.

We now show that these constants are uniquely determined by the requirement that ψ satisfy (2.32). Substituting (2.35) into L we obtain

$$L(\psi) = c_0L(x^{j+m}e^{ax}) + c_1L(x^{j+m-1}e^{ax}) + \cdots + c_mL(x^je^{ax}). \tag{2.36}$$

The terms in this sum can be computed using (2.34). We note that

$$p(a) = p'(a) = \cdots = p^{(j-1)}(a) = 0, \quad p^{(j)} \neq 0,$$

since a is a root of p with multiplicity j. Thus, if $k \geq j$,

$$L(x^k e^{ax}) = \left[\binom{k}{k-j}\ p^{(j)}(a)x^{k-j} + \binom{k}{k-j-1}\ p^{(j+1)}(a)x^{k-j-1} + \cdots + p^{(k)}(a)\right]\ e^{ax}.$$

Then we have

$$L(x^{j+m}e^{ax}) = \left[\binom{j+m}{m}\ p^{(j)}(a)x^m + \binom{j+m}{m-1}\ p^{(j+1)}(a)x^{m-1} + \cdots + p^{(j+m)}(a)\right]\ e^{ax}$$

$$L(x^{j+m-1}e^{ax}) = \left[\binom{j+m-1}{m-1}\ p^{(j)}(a)x^{m-1} + \binom{j+m-1}{m-2}\ p^{(j+1)}(a)x^{m-2} + \cdots + p^{(j+m-1)}(a)\right]\ e^{ax}$$

$$\vdots$$

$$L(x^j e^{ax}) = \binom{j}{0}\ p^{(j)}(a)e^{ax}.$$

Using these computations in (2.36) and noting that (2.33), we see that ψ satisfies (2.32) if and only if

$$c_0\ \binom{j+m}{m}\ p^{(j)}(a) = b_0,$$

$$c_0\ \binom{j+m}{m-1}\ p^{(j+1)}(a) + c_1\ \binom{j+m-1}{m-1}\ p^{(j)}(a) = b_1,$$

$$\vdots$$

$$c_0 p^{(j+m)}(a) + c_1 p^{(j+m-1)}(a) + \cdots + c_m p^{(j)}(a) = b_m$$

This is a set of $m+1$ linear equations for the constants $c_0, c_1, \cdots, c_m$. They have a unique solution, which can be obtained by solving the equations in succession since $p^(j)(a) \neq 0$. Alternately, we see that the determinant of the coefficients is just

$$\binom{j+m}{m}\ \binom{j+m-1}{m-1} \cdots 1\ \left[p^{(j)}(a)\right]^{m+1} \neq 0.$$

This completes the proof of the theorem. □

The Justification of the annihilator method when the right side of $L(y) = b(x)$ is the sum of terms of the form $P(x)e^{ax}$ can be reduced to Theorem 2.23, by noting that if ψ_1, ψ_2 satisfy

$$L(\psi_1) = b_1, \qquad L(\psi_2) = b_2,$$

respectively, then $\psi_1 + \psi_2$ satisfies

$$L(\psi_1 + \psi_2) = b_1 + b_2.$$

Chapter 3

Linear equations with variable coefficients

3.1 Introduction

A linear differential equation of order n with variable coefficients is an equation of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x),$$

where $a_0, a_1, \cdots, a_n, b$ are complex-valued functions on some real interval I. Points where $a_0(x) = 0$ are called *singular points*, and often the equation requires special consideration at such points. Therefore in this chapter we assume that $a_0(x) \neq 0$ on I. By dividing by a_0, We can obtain an equation of the same form, but with a_0 replaced by the constant 1. Thus we consider the equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x) \tag{3.1}$$

As in the case when $a_1, a_2, \cdots, a_n$, are constants we designate the left side of (3.1) by $L(y)$. Thus

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y \tag{3.2}$$

and (3.1) becomes simply $L(y) = b(x)$. If $b(x) = 0$ for all x on I we say $L(y) = 0$ is a *homogeneous equation*, whereas if $b(x) \neq 0$ for some x in I, the equation $L(y) = b(x)$ is called a *non-homogeneous equation.*

We give a meaning to L itself as an operator which takes each function ϕ, which has

n derivatives on I, into the function $L(\phi)$ on I whose value at x is given by

$$L(\phi)(x) = \phi^{(n)}(x) + a_1\phi^{(n-1)}(x) + \cdots + a_n\phi(x).$$

Thus a solution of (3.1) on I is a function ϕ on I which has n derivatives there, and which satisfies $L(\phi) = b$.

We assume that the complex-valued functions $a_1, \cdots, a_n, b$ are continuous on some real interval I. and $L(y)$ will always denote the expresion (3.2).

3.2 Initial value problem for the homogeneous equation

Theorem 3.1. (Existence Thorem) Let $a_1, \cdots, a_n$ be continuous functions on an interval I containing the point x_0. If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are any n constants, there erists a solution ϕ of

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0$$

on I satisfying

$$\phi(x_0) = \alpha_1, \quad \phi'(x_0) = \alpha_2, \quad \cdots, \quad \phi^{(n-1)}(x_0) = \alpha_n.$$

We stress two things about this theorem :
(i) the solution exists on the entire interval I where $a_1, a_2, \cdots, a_n$ are continuous, and
(ii) every initial value problem has a solution.

Neither of these results may be true if the coefficient of $y^{(n)}$ vanishes somewhere in I.

For example, consider the equation

$$xy' + y = 0,$$

whose coefficients are continuous for all real x. This equation and the initial condition $y(1) = 1$ has the solution ϕ_1, where

$$\phi_1(x) = \frac{1}{x}.$$

But this solution exists only for $0 < x < \infty$. Also, if ϕ is any solution, then

$$x\phi(x) = c,$$

where c is some constant. Thus only the trivial solution ($c = 0$) exists at the origin, which implies that the only initial value problem

$$xy' + y = 0, \quad y(0) = \alpha_1,$$

which has a solution is the one for which $\alpha_1 = 0$.

Just as in the case where the coefficients $a_j, \quad (j = 1, \cdots, n)$ are constants, the uniqueness of the solution ϕ given in Theorern 3.1 is demonstrated with the aid of an estimate for

$$\phi(x) = \left[|\phi(x)|^2 + |\phi'(x)|^2 + \cdots + \left|\phi^{(n-1)}(x)\right|^2 \right]^{1/2},$$

Remark 3.2. If I is a closed bounded interval, that is, of the form $a \leq x \leq b$ with a, b real, and if the a_j are continuous on I, then there always exiut finite constants by such that $|a_j(x)| \leq b_j$, on I.

Theorem 3.3. *Let $b_1, \cdots, b_n$, be non-negative constants such that for all x in I.*

$$|a_j(x)| \leq b_j \ , \qquad (j = 1, 2, \cdots, n)$$

and define k by

$$k = 1 + b_1 + b_2 + \cdots + b_n.$$

If x_0 is a point in I, and ϕ is a solution of $L(y) = 0$ on I, then

$$\|\phi(x_0)\| \ e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\| \ e^{k|x-x_0|} \tag{3.3}$$

for all x in I.

Proof. Since $L(\phi) = 0$ we have

$$\phi^{(n)} = -a_1(x)\phi^{(n-1)}(x) - \cdots - a_n(x)\phi(x),$$

and therefore

$$\begin{aligned} |\phi^{(n)}| &= |a_1(x)\phi^{(n-1)}(x) + \cdots + a_n(x)\phi(x)| \\ &\leq |a_1(x)||\phi^{(n-1)}(x)| + cdots + |a_n(x)||\phi(x)| \\ &\leq b_1|\phi^{(n-1)}(x)| + cdots + b_n|\phi(x)| \end{aligned}$$

Hence

$$|\phi^{(n)}| \leq b_1|\phi^{(n-1)}(x)| + cdots + b_n|\phi(x)| \tag{3.4}$$

Now letting $u(x) = \|\phi(x)\|^2$ for $x \in I$. Then

$$\begin{aligned}
u(x) &= |\phi(x)|^2 + |\phi'(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2 \\
&= \phi(x)\overline{\phi(x)} + \phi'(x)\overline{\phi'(x)} + \cdots + \phi^{(n-1)}(x)\overline{\phi^{(n-1)}(x)}, \text{ since } |z|^2 = z\overline{z} \\
&= \phi(x)\overline{\phi}(x) + \phi'(x)\overline{\phi'}(x) + \cdots + \phi^{(n-1)}(x)\overline{\phi^{(n-1)}}(x)
\end{aligned}$$

Then

$$\begin{aligned}
u'(x) &= \phi(x)\overline{\phi'}(x) + \phi'(x)\overline{\phi}(x) + \phi'(x)\overline{\phi''}(x) + \phi''(x)\overline{\phi'}(x) \\
&\quad + \cdots + \phi^{(n)}(x)\overline{\phi^{(n-1)}}(x) + \phi^{(n-1)}(x)\overline{\phi^{(n)}}(x) \\
|u'(x)| &= |\phi(x)\overline{\phi'}(x) + \phi'(x)\overline{\phi}(x) + \phi'(x)\overline{\phi''}(x) + \phi''(x)\overline{\phi'}(x) \\
&\quad + \cdots + \phi^{(n)}(x)\overline{\phi^{(n-1)}}(x) + \phi^{(n-1)}(x)\overline{\phi^{(n)}}(x)| \\
&\leq |\phi(x)\overline{\phi'}(x)| + |\phi'(x)\overline{\phi}(x)| + |\phi'(x)\overline{\phi''}(x)| + |\phi''(x)\overline{\phi'}(x)| \\
&\quad + \cdots + |\phi^{(n)}(x)\overline{\phi^{(n-1)}}(x)| + |\phi^{(n-1)}(x)\overline{\phi^{(n)}}(x)| \\
&= |\phi(x)||\overline{\phi'}(x)| + |\phi'(x)||\overline{\phi}(x)| + |\phi'(x)||\overline{\phi''}(x)| + |\phi''(x)||\overline{\phi'}(x)| \\
&\quad + \cdots + |\phi^{(n)}(x)||\overline{\phi^{(n-1)}}(x)| + |\phi^{(n-1)}(x)||\overline{\phi^{(n)}}(x)| \\
&= |\phi(x)||\phi'(x)| + |\phi'(x)||\phi(x)| + |\phi'(x)||\phi''(x)| + |\phi''(x)||\phi'(x)| \\
&\quad + \cdots + |\phi^{(n)}(x)||\phi^{(n-1)}(x)| + |\phi^{(n-1)}(x)||\phi^{(n)}(x)| \\
&= 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||\phi''(x)| + \cdots + 2|\phi^{(n-1)}(x)||\phi^{(n)}(x)| \\
&\leq 2\ |\phi(x)||\phi'(x)| + 2\ |\phi'(x)||\phi''(x)| + \cdots + \\
&\quad 2|\phi^{(n-1)}(x)|\ \left(|a_1(x)||\phi^{(n-1)}(x)| + |a_2(x)||\phi^{(n-2)}(x)| + \cdots + |a_n(x)||\phi(x)|\right) \\
&\leq (1 + |b_n|)|\phi(x)|^2 + (2 + |b^{(n-1)}|)|\phi'(x)|^2 + \cdots + \\
&\quad (2 + |b_1|)|\phi^{(n-2)}(x)|^2 + (1 + 2|b_1| + |b_2| + \cdots + |b_n|)|\phi^{(n-1)}(x)|^2, \quad \text{using(3.4)} \\
&\leq 2(1 + |b_1| + |b_2| + \cdots + |b_n|)|\phi(x)|^2 + 2(1 + |b_1| + |b_2| + \cdots + |b_n|)|\phi'(x)|^2 \\
&\quad + \cdots + 2(1 + |b_1| + |b_2| + \cdots + |b_n|)|\phi^{(n-2)}(x)|^2 \\
&\quad + 2(1 + |b_1| + |b_2| + \cdots + |b_n|)|\phi^{(n-1)}(x)|^2 \\
&= 2(1 + |b_1| + |b_2| + \cdots + |b_n|)\ \left(|\phi(x)|^2 + |\phi'(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2\right) \\
&= 2\ k\ u(x), \quad \text{where } k = 1 + |b_1| + |b_2| + \cdots + |b_n|
\end{aligned}$$

Therefore $|u'(x)| \leq 2\ k\ u(x)$. That is $-2\ k\ u(x) \leq u'(x) \leq 2\ k\ u(x)$.

Take $u'(x) \leq 2\ k\ u(x)$. Then

$$\begin{aligned} u'(x) - 2\ k\ u(x) &\leq 0 \\ e^{-2kx}u'(x) + u(x)(-2ke^{-2kx}) &\leq 0 \\ (e^{-2kx}\ u(x))' &\leq 0 \end{aligned}$$

Let $x_0 < x$

$$\int_{x_0}^{x} (e^{-2kt}\ u(t))'\ dt \leq 0$$

$$\begin{aligned} e^{-2kx}\ u(x) - e^{-2kx_0}\ u(x_0) &\leq 0 \\ e^{-2kx}\ u(x) &\leq e^{-2kx_0}\ u(x_0) \\ u(x) &\leq e^{2k(x-x_0)}\ u(x_0) \\ \|\phi(x)\|^2 &\leq \|\phi(x_0)\|^2\ e^{2k(x-x_0)} \\ \|\phi(x)\| &\leq \|\phi(x_0)\|\ e^{k(x-x_0)} \end{aligned}$$

Similarly taking $-2\ k\ u(x) \leq u'(x)$ we can show that $\|\phi(x_0)\|\ e^{-k(x-x_0)} \leq \|\phi(x)\|$. Hence

$$\|\phi(x_0)\|\ e^{-k(x-x_0)} \leq \|\phi(x)\| \leq \|\phi(x_0)\|\ e^{k(x-x_0)} \quad \text{when } x_0 < x \tag{3.5}$$

In a similar way, we can show that

$$\|\phi(x_0)\|\ e^{-k(x_0-x)} \leq \|\phi(x)\| \leq \|\phi(x_0)\|\ e^{k(x_0-x)} \quad \text{when } x < x_0 \tag{3.6}$$

Hence from (3.5) and (3.6), we have

$$\|\phi(x_0)\|\ e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\|\ e^{k|x-x_0|}$$

where $k = 1 + |b_1| + |b_2| + \cdots + |b_n|$. Hence the theorem. □

Theorem 3.4. (Uniqueness Theorem) Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be any n constants, and let x_0 be any real number. On any interval I containing x_0 there exists at most one solution ϕ of $L(y) = 0$ on I satisfying

$$\phi(x_0) = \alpha_1, \quad \phi'(x_0) = \alpha_2, \quad \cdots, \quad \phi^{(n-1)}(x_0) = \alpha_n.$$

Proof. Suppose ϕ and ψ are two solutions of the initial value problem $L(y) = 0,\ y(x_0) = \alpha_1,\ y'(x_0) = \alpha_2,\ \cdots,\ y^{(n-1)}(x_0) = \alpha_n$ on I. Then we have to prove that $\phi(x) = \psi(x)$ for all x. Let $\chi = \phi(x) - \psi(x)$. Even though the functions

a_j are continuous on I they need not be bounded there. However, Let x be any point on I other than x_0. Let J be any closed bounded interval in I which contains x, x_0. On this interval the functions a_j are bounded. That is $|a_j(x)| \leq b_j$ on J for $j = 1, 2, \cdots, n$.
Then $L(\chi) = L(\phi) - L(\psi) = 0$, and $\chi(x_0) = 0,\ \chi'(x_0) = 0,\ \cdots,\ \chi^{(n-1)}(x_0) = 0$. Then $\|\chi(x_0)\| = 0$, and applying the inequality (3.3) to χ we see that $\|\chi(x)\| = 0$ for all x in J. This implies that $\chi(x) = 0$ for all x in J. Since x was chosen to be any point in I other than x_0, we have $\phi(x) = \psi(x)$ for all x in I. □

3.3 Solution of the homogeneous equation

If $\phi_1, \phi_2, \cdots, \phi_m$ are any m solutions of the $n-$th order equation $L(y) = 0$ on an interval I, and $c_1, \cdots, c_m$ are any m constants, then

$$L(c_1\phi_1 + \cdots + c_m\phi_m = c_1 L(\phi_1 + \cdots + c_m L(\phi_m),$$

which implies that $c_1\phi_1 + \cdots + c_m\phi_m$ is also a solution. In words, any linear combination of solutions is again a solution. The trivial solution is the function which is identically zero on I.

As in the case of an L with constant coefficients, every solution of $L(y) = 0$ is a linear combination of any n linearly independent solutions. Recall that n functions $\phi_1, \cdots, \phi_n$ defined on an interval I are said to be *linearly independent* if the only constants $c_1, \cdots, c_n$ such that

$$c_1\phi_1(x) + \cdots + c_n\phi_n(x) = 0$$

for all x in I are the constants

$$c_1 = c_2 = \cdots + c_n = 0.$$

we construct n linearly independent solutions, and show that every solution is a linear combination of these. we show that every solution is a linear combination of any n linearly independent solutions.

Theorem 3.5. *There exist n linearly independent solutions of $L(y) = 0$ on I.*

Proof. Let us consider the initial value problem

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

with initial condition

$$y(x_0) = 1,\ y'(x_0) = 0,\ \cdots,\ y^{(n-1)}(x_)= 0$$

Then by existence and uniqueness theorem, the above initial value problem has a unique solution. Let it be ϕ_1. Then

$$L(\phi_1) = 0,\ \phi_1(x_0) = 1,\ \phi_1'(x_0) = 0,\ \cdots,\ \phi_1^{(n-1)}(x_) = 0.$$

Now let us consider another initial value problem

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

with initial condition

$$y(x_0) = 0,\ y'(x_0) = 1,\ \cdots,\ y^{(n-1)}(x_) = 0$$

Then again by existence and uniqueness theorem, the above initial value problem has a unique solution. Let it be ϕ_2. Then

$$L(\phi_2) = 0,\ \phi_2(x_0) = 0,\ \phi_2'(x_0) = 1,\ \cdots,\ \phi_2^{(n-1)}(x_) = 0.$$

Continuing in this manner after n steps we get n functions $\phi_1, \phi_2, \cdots, \phi_n$ solutions of $L(y) = 0$ satisfying

$$\begin{aligned} \phi_i^{(i-1)}(x_0) = 1 \quad & \text{for all } i = 1, 2, \cdots, n \\ \phi_i^{(j-1)}(x_0) = 0 \quad & \text{for all } j = 1, 2, \cdots, n,\ j \neq i \end{aligned} \tag{3.7}$$

Now let us show that $\{\phi_1, \phi_2, \cdots, \phi_n\}$ is linearly independent.

Suppose there are constants $c_1, c_2, \cdots, c_n$ such that $c_1\phi_1 + \cdots + c_n\phi_n = 0$

for all x in I. Differentiating we see that

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0 \tag{3.8}$$

for all x in I. Then also

$$\begin{aligned} c_1\phi_1'(x) + c_2\phi_2'(x) + \cdots + c_n\phi_n'(x) = 0 \\ c_1\phi_1''(x) + c_2\phi_2''(x) + \cdots + c_n\phi_n''(x) = 0 \\ \vdots \\ c_1\phi_1^{(n-1)}(x) + c_2\phi_2^{(n-1)}(x) + \cdots + c_n\phi_n^{(n-1)}(x) = 0 \end{aligned} \tag{3.9}$$

for all x in I. In particular, the equations (3.8) and (3.9) must hold at x_0. Putting $x = x_0$ in (3.8) we find, using (3.7), that $c_1\ (1) + c_2\ (0) + \cdots + c_n\ (0) = 0$, or $c_1 = 0$. Putting $x = x_0$ in the equations (3.9) we obtain $c_2 = c_3 = \cdots + c_n = 0$ and thus the solutions $\phi_1, \cdots, \phi_n$ are linearly independent. $\square$

Theorem 3.6. *Let $\phi_1, \cdots, \phi_n$ be the n solutions of $L(y) = 0$ on I satisfying (3.7). If ϕ is any solution of $L(y) = 0$ on I, there are n constants $c_1, \cdots, c_n$ such that*

$$\phi = c_1\phi_1 + \cdots + c_n\phi_n.$$

Proof. Let ϕ be a solution of $L(y) = 0$ on I. Let $\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2 \cdots, \phi^{(n-1)}(x_0) = \alpha_n$.

Consider the function

$$\psi = \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_n\phi_n.$$

Then $L(\psi) = \alpha_1 L(\phi_1) + \alpha_2 L(\phi_2) + \cdots + \alpha_n L(\phi_n) = 0$ and so ψ is a solution of $L(y) = 0$ and clearly

$$\psi(x_0) = \alpha_1\phi_1(x_0) + \alpha_2\phi_2(x_0) + \cdots + \alpha_n\phi_n(x_0) = \alpha_1,$$

since $\phi_1(x_0) = 1,\ \phi_2(x_0) = 0, \cdots,\ \phi_n(x_0) = 0.$

Similarly using the other relations in (3.7) we see that

$$\psi(x_0) = \alpha_1,\ \psi'(x_0) = \alpha_2, \cdots, \psi^{(n-1)}(x_0) = \alpha_n.$$

Thus *psi* is a solution of $L(y) = 0$ having the same initial conditions at x_0 as ϕ. By uniqueness Theorem, we must have $\phi = \psi$, that is

$$\phi = \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_n\phi_n.$$

We have proved the theorem with constants $c_1 = \alpha_1, c_2 = \alpha_2, \cdots, c_n = \alpha_n.$ $\square$

Remark 3.7. A set of functions which has the property that, if ϕ_1, ϕ_2 belong to the set, and c_1, c_2 are any two constants, then $c_1\phi_1 + c_2\phi_2$ belongs to the set also is called a *linear space* of functions. We have just seen that the set of all solutions of $L(y) = 0$ on an interval I is a linear space of functions. If a linear space of functions contains n functions $\phi_1, \phi_2, \cdots, \phi_n$, which are linearly independent and such that every function in the space can be represented as a linear combination of these, then $\phi_1, \cdots, \phi_n$ is called a *basis* for the linear space, and the *dimension* of the linear space is the integer n. Then the functions $\phi_1, \cdots, \phi_n$ satisfying the initial conditions (3.7) form a basis for the solutions of $L(y) = 0$ on I, and this linear space of functions has dimension n.

3.4 The Wronskian and linearly independent

In order to show that any set of n linearly independent solutions of $L(y) = 0$ can serve as a basis for the solutions of $L(y) = 0$, we consider the Wronskian $W(\phi_1, \phi_2, \cdots, \phi_n)$ of any n solutions $\phi_1, \phi_2, \cdots, \phi_n$.

Definition 3.8. The wronskian $W(\phi_1, \phi_2, \cdots, \phi_n)$ of n functions $\phi_1, \phi_2, \cdots, \phi_n$ having $n-1$ derivatives on an interval I is defined to be the determinant function

$$W(\phi_1, \phi_2, \cdots, \phi_n) = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_n' \\ \vdots & & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}$$

Theorem 3.9. *If $\phi_1, \phi_2, \cdots, \phi_n$ are solutions of $L(y) = 0$ on an interval I, then they are linearly independent there if and only if $W(\phi, \phi_2, \cdots, \phi_n)(x) \neq 0$ for all x in I.*

Proof. First suppose $W(\phi_1, \phi_2, \cdots, \phi_n)(x) \neq 0$ for all x in I, and let $c_1, c_2, \cdots, c_n$ be constants such that

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0 \tag{3.10}$$

for all x in I. Then also

$$\begin{aligned} c_1\phi_1'(x) + c_2\phi_2'(x) + \cdots + c_n\phi_n'(x) &= 0 \\ c_1\phi_1''(x) + c_2\phi_2''(x) + \cdots + c_n\phi_n''(x) &= 0 \\ &\vdots \\ c_1\phi_1^{(n-1)}(x) + c_2\phi_2^{(n-1)}(x) + \cdots + c_n\phi_n^{(n-1)}(x) &= 0 \end{aligned} \tag{3.11}$$

for all x in I. For a fixed x the equations (3.10), (3.11) are linear homogeneous equations satisfied by $c_1, c_2, \cdots, c_n$. Hence the matrix representation of the equations (3.10) and (3.11) is

$$\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since the determinant of the coefficients of $c_1, c_2 \cdots, c_n$ in (3.10) and (3.11) is just $W(\phi_1, \phi_2, \cdots, \phi_n)(x)$ which is not zero. Therefore the matrix

$$\begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix}$$

is non-singular. Hence the above matrix equation has unique solution namely $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. That is $c_1 = 0,\ c_2 = 0, \cdots, c_n = 0$. Therefore $c_1 = 0,\ c_2 = 0, \cdots, c_n = 0$ is the only solution of (3.10) and (3.11). This proves that $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I.

Conversely, assume ϕ_1, ϕ_2 are linearly independent on I. Suppose that there is an x_0 in I such that $W(\phi_1, \phi_2, \cdots, \phi_n)(x_0) = 0$. This implies that the system of linear equations

$$\begin{aligned} c_1\phi_1(x_0) + c_2\phi_2(x_0) + \cdots + c_n\phi_n(x_0) &= 0 \\ c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \cdots + c_n\phi_n'(x_0) &= 0 \\ c_1\phi_1''(x_0) + c_2\phi_2''(x_0) + \cdots + c_n\phi_n''(x_0) &= 0 \\ &\vdots \\ c_1\phi_1^{(n-1)}(x_0) + c_2\phi_2^{(n-1)}(x_0) + \cdots + c_n\phi_n^{(n-1)}(x_0) &= 0 \end{aligned} \tag{3.12}$$

has a solution $c_1, c_2, \cdots, c_n$, where at least one of these numbers is not zero. Let $c_1, c_2, \cdots, c_n$ be such a solution and consider the function $\psi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$. Now $L(\psi) = 0$, and from (3.11) we see that

$$\psi(x_0) = 0,\ \psi'(x_0) = 0, \cdots, \psi^{(n-1)}(x_0) = 0.$$

From the Uniqueness theorem (Theorem 3.4), we infer that $\psi(x) = 0$ for all x in I

and thus

$$c_1\phi(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0$$

for all x in I. But this contradicts the fact that $\phi_1, \phi_2 \cdots, \phi_n$ are linearly independent on I. Thus the superposition that there was a point x_0 in I such that $W(\phi_1, \phi_2, \cdots, \phi_n) = 0$ must be false. We have consequently proved that $W(\phi_1, \phi_2, \cdots, \phi_n) \neq 0$ for all x in I. □

Theorem 3.10. *Let $\phi_1, \phi_2, \cdots, \phi_n$ be n linearly independent solutions of $L(y) = 0$ on an interval I. If ϕ is any solution of $L(y) = 0$ on I, it can be represented in the form*

$$\phi = c_1\phi_1 + \cdots + c_n\phi_n,$$

where $c_1, c_2, \cdots, c_n$ are constants. Thus any set of n linearly independent solutions of $L(y) = 0$ on I is a basis for tlhe solutions of $L(y) = 0$ on I.

Proof. Let x_0 be a point in I, and suppose

$$\phi(x_0) = \alpha_1,\ \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n.$$

We show that there exist unique constants $c_1, c_2, \cdots, c_n$ such that

$$\psi = c_1\phi_1 + \cdots + c_n\phi_n$$

is a solution of $L(y) = 0$ satisfying

$$\psi(x_0) = \alpha_1,\ \psi'(x_0) = \alpha_2, \cdots, \psi^{(n-1)}(x_0) = \alpha_n.$$

By the uniqueness result Theorem 3.4 we then have $\phi = \psi$, or

$$\phi = c_1\phi_1 + \cdots + c_n\phi_n.$$

The initisl conditions for ψ are equivalent to the following equations for $c_1, \cdots, c_n$:

$$\begin{aligned} c_1\phi_1(x_0) + \cdots + c_n\phi_n(x_0) &= \alpha_1 \\ c_1\phi_1'(x_0) + \cdots + c_n\phi_n'(x_0) &= \alpha_2 \\ &\vdots \\ c_1\phi_1^{(n-1)}(x_0) + \cdots + c_n\phi_n^{(n-1)}(x_0) &= \alpha_n \end{aligned} \tag{3.13}$$

This is a set of n linear equations for $c_1, \cdots, c_n$. The determinant of the coefficients is $W(\phi_1, \cdots, \phi_n)(x_0)$, which is not zero since $\phi_1, \cdots, \phi_n$ are linearly independent (Theorem 3.9). Therefore there is a unique solution $c_1, \cdots, c_n$ of the equations (3.13), and this completes the proof. □

Theorem 3.11. *If $\phi_1, \phi_2, \cdots, \phi_n$ are two solutions of $L(y) = 0$ on an interval I containing a point x_0, then*

$$W(\phi_1, \phi_2, \cdots, \phi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\ dt\right] W(\phi_1, \phi_2, \cdots, \phi_n)(x_0). \quad (3.14)$$

Proof. We first prove the result for the case $n = 2$ and then give a proof which is valid for general n.

Case 1: $n = 2$

In this case $W = \phi_1\phi_2' - \phi_1'\phi_2$, and therefore

$$\begin{aligned} W' &= \phi_1'\phi_2' + \phi_1\phi_2'' - \phi_1''\phi_2 - \phi_1'\phi_2' \\ &= \phi_1\phi_2'' - \phi_1''\phi_2 \end{aligned}$$

Let ϕ_1, ϕ_2 be two solutions of $L(y) = 0$. Then we have

$$\phi_1'' + a_1\phi_1' + a_2\phi_1 = 0 \text{ and } \phi_2'' + a_1\phi_2' + a_2\phi_2 = 0$$

Thus

$$\begin{aligned} W'(\phi_1, \phi_2) &= \phi_1(-a_1\phi_2' - a_2\phi_2) - (-a_1\phi_1' - a_2\phi_1)\phi_2) \\ &= -W_1(\phi_1\phi_2' - \phi_1'\phi_2) \\ &= -a_1 W(\phi_1, \phi_2) \end{aligned}$$

we see that $W(\phi_1, \phi_2)$ satisfies the linear first order equation $y' + a_1(x)y = 0$.

That is $W' + a_1 W = 0$.

Hence $W(\phi_1, \phi_2)(x) = c\exp\left[-\int_{x_0}^{x} a_1(t)\ dt\right]$, where c is some constant. By putting $x = x_0$ we obtain $c = W(\phi_1, \phi_2)(x_0)$,and thus

$$W(\phi_1, \phi_2)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\ dt\right] W(\phi_1, \phi_2)(x_0).$$

Case 2: For general n

Let $\phi_1, \phi_2, \cdots, \phi_n$ be n solutions of $L(y) = 0$. Let $W = W(\phi_1, \phi_2, \cdots, \phi_n)$. From the definition of W, its derivatives W' is the sum of n determinants. That is, $W' = V_1 + V_2 + \cdots + V_n$, where V_k differ from W only in its $k-$throw and $k-$th row of V_k is obtained by differentiating $k-$th row of W. Thus

$$W' = \begin{vmatrix} \phi_1' & \phi_2' & \cdots & \phi_n' \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1'' & \phi_2'' & \cdots & \phi_n'' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} + \cdots + \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-2)} & \phi_2^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ \phi_1^{(n)} & \phi_2^{(n)} & \cdots & \phi_n^{(n)} \end{vmatrix}$$

The first $n-1$ determinants $V_1, V_2, \cdots, V_{n-1}$ are all zero, since they each have two identical rows. Since $\phi_1, \phi_2, \cdots, \phi_n$ are solutions of $L(y) = 0$, we have

$$\phi_i^{(n)} = -a_1\phi_i^{(n-1)} - a_2\phi_i^{(n-2)} - \cdots - a_n\phi_1.$$

Therefore,

$$W' = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ -\sum_{j=0}^{n-1} a_{n-j}\phi_1^{(j)} & -\sum_{j=0}^{n-1} a_{n-j}\phi_2^{(j)} & \cdots & -\sum_{j=0}^{n-1} a_{n-j}\phi_n^{(j)} \end{vmatrix}$$

The value of this determinant is unchanged if we multiply any row by a number and add to the last row. Hence

$$W' = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ -a_1\phi_1^{(n-1)} & -a_1\phi_2^{(n-1)} & \cdots & -a_1\phi_n^{(n-1)} \end{vmatrix}$$

$$= -a_1 \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} = -a_1\, W$$

Thus W satisfies the first order equation $W' + a_1W = 0$.

Hence $W(x) = c \exp \left[-\int_{x_0}^{x} a_1(t)\ dt \right]$, where c is some constant. Setting $x = x_0$ we see that $c = W(x_0)$, and thus

$$W(x) = \exp \left[-\int_{x_0}^{x} a_1(t)\ dt \right] W(x_0),$$

That is $W(\phi_1, \phi_2, \cdots, \phi_n)(x) = \exp \left[-\int_{x_0}^{x} a_1(t)\ dt \right] W(\phi_1, \phi_2, \cdots, \phi_n)(x_0)$, □

Corollary 3.12. *If the coefficients a_k of L are constants, then*

$$W(\phi_1, \phi_2, \cdots, \phi_n)(x) = e^{-a_1(x-x_0)}\ W(\phi_1, \phi_2, \cdots, \phi_n)(x_0)$$

3.5 Reduction of the order of a homogeneous equation

Suppose we have found by some means one solution ϕ_1 of the equation

$$L(y) = y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = 0.$$

It is then possible to take advantage of this information to reduce the order of the equation to be solved by one. The idea is the same one employed in the variation of constants method. We try to find solutions ϕ of $L(y) = 0$ of the form $\phi = u\phi_1$, where u is some function.

Theorem 3.13. *Let ϕ_1 be a solution of $L(y) = 0$ on an interval I, and suppose $\phi_1(x) \neq 0$ on I. If $v_2, v_3, \cdots, v_n$ is any basis on I for the solutions of the linear equation in v of order $n-1$, and if $v_k = u_k'$, $(k = 2, \cdots, n)$ then $\phi_1, u_2\phi_1, \cdots, u_n\phi_1$ is a basis for the solutions of $L(y) = 0$ on I.*

Proof. Let $\phi_1(x) \neq 0$ be a solution of $L(y) = 0$ on an interval I. Let u be a function on I such that $\phi = u\phi_1$ is a solution of $L(y) = 0$. Then we have

$$\begin{aligned} 0 &= (u\phi_1)^{(n)} + a_1(x)(u\phi_1)^{(n-1)} + \cdots + a_{n-1}(x)(u\phi_1)' + a_n(x)(u\phi_1) \\ &= u^{(n)}\phi_1 + \cdots + u\phi_1^{(n)} + a_1 u^{(n-1)}\phi_1 + \cdots + a_1 u\phi_1^{(n-1)} + \cdots \\ &\quad + a_{n-1}u'\phi_1 + a_{n-1}u\phi_1' + a_n u\phi_1. \end{aligned}$$

The coefficient of u in this equation is just $L(\phi_1) = 0$. Therefore, if $v = u'$, this is a linear equation of order $n-1$ in v,

$$\phi_1 v^{(n-1)} + \cdots + \left[n\phi_1^{(n-1)} + a_1(n-1)\phi_1^{(n-2)} + \cdots + a_{n-1}\phi_1 \right] v = 0 \tag{3.15}$$

The coefficient of $v^{(n-1)}$ is ϕ_1, and hence if $\phi_1(x) \neq 0$ on an interval I this equation has $n-1$ linearly independent solutions $v_2, v_3, \cdots, v_n$ on I. If x_0 is some point in I and

$$u_k = \int_{x_0}^{x} v_k(t)\ dt, \qquad k = 2, 3, \cdots, n$$

Then we have $u_k' = v_k$, and the functions

$$\phi_1, u_2\phi_1, \cdots, u_n\phi_1 \tag{3.16}$$

are solutions of $L(y) = 0$. Moreover these functions form a basis for the solutions of $L(y) = 0$ on I. For suppose we have constants $c_1, c_2, \cdots, c_n$ such that

$$c_1\phi_1 + c_2u_2\phi_1 + \cdots + c_nu_n\phi_1 = 0.$$

Since $\phi_1(x) \neq 0$ on I this implies

$$c_1 + c_2u_2 + \cdots + c_nu_n = 0, \tag{3.17}$$

and differentiating we obtain

$$c_2u_2' + c_3u_3' + \cdots + c_nu_n' = 0, \quad \text{or} \quad c_2v_2 + c_3v_3 + \cdots + c_nv_n = 0.$$

Since $v_2, v_3, \cdots, v_n$ are linearly independent on I we have

$$c_2 = c_3 = \cdots = c_n = 0.$$

and from (3.17) we obtain $c_1 = 0$ also. Thus the functions in (3.16) form a basis for the solutions of $L(y) = 0$ on I. □

Theorem 3.14. *If ϕ_1 is a solution of $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$ on an interval I, and $\phi_1(x) \neq 0$ on I, a second solution ϕ_2 of $L(y) = 0$ on I is given by*

$$\phi_2(x) = \phi_1(x) \int_{x_0}^{x} \frac{1}{[\phi_1(s)]^2} \exp\left[-\int_{x_0}^{x} a_1(t)\ dt \right] ds.$$

The functions ϕ_1, ϕ_2 form a basis for the solutions of $L(y) = 0$ on I.

Proof. Let $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$ and if ϕ_1 is a solution on I we have

$$\begin{aligned} L(u\phi_1) &= (u\phi_1)'' + a_1(u\phi_1)' + a_2(u\phi_1) \\ &= u''\phi_1 + 2u'\phi_1' + u\phi_1'' + a_1u'\phi_1 + a_1u\phi_1' + a_2u\phi_1 \\ &= u''\phi_1 + u'(2\phi_1' + a_1\phi_1). \end{aligned}$$

Thus if $v = u'$ and u is such that $L(u\phi_1) = 0$,

$$\phi_1 v' + (2\phi_1' + a_1\phi_1)v = 0. \tag{3.18}$$

But (3.18) is a linear equation of order one, and can always be solved explicitly provided $\phi_1(x) \neq 0$ on I. Indeed v satisfies

$$\phi_1^2 v' + (2\phi_1\phi_1' + a_1\phi_1^2)v = 0, \tag{3.19}$$

which is just (3.18) multiplied by ϕ_1. Thus

$$(\phi_1^2 v)' + a_1(\phi_1^2 v) = 0,$$

which implies that

$$\phi_1^2(x)v(x) = c\exp\left[-\int_{x_0}^{x} a_1(t)\ dt\right],$$

where x_0 is a point in I and c is a constant. Since any constant multiple of a solution of (3.19) is again a solution, we see that

$$v(x) = \frac{1}{[\phi_1(s)]^2}\ \exp\left[-\int_{x_0}^{x} a_1(t)\ dt\right] ds.$$

is a solution of (3.19), and also of (3.18). Therefore two independent solutions of $L(y) = 0$ on I are ϕ_1 and ϕ_2 where

$$\phi_2(x) = \phi_1(x)\int_{x_0}^{x} \frac{1}{[\phi_1(s)]^2}\ \exp\left[-\int_{x_0}^{x} a_1(t)\ dt\right] ds.$$

□

Example 3.15. Consider the equation $y'' - \dfrac{2}{x^2}y = 0, \qquad (0 < x < \infty).$

Let $\phi_(x) = x^2$. Then it can be easily verified that the ϕ_1 is a solution on $0 < x < \infty$ and since the function does not vanish on this interval there is another independent solution ϕ_2 of the form $\phi_2 = u\phi_1$. If $v = u'$ we find that v satisfies

$$x^2v' + 4xv = 0 \quad \text{or} \quad xv' + 4v = 0.$$

Then $v' + \dfrac{4}{x} = 0$. This is a linear first order equation. A solution for this is given by $v(x) = x^{-4}, \quad (0 < x < \infty)$, and therefore $u' = v = x^{-4}$.

Hence $u(x) = \int v(x)dx = \frac{-1}{3x^3}$ and so $\phi_2(x) = u(x)\phi_1(x) = \frac{-1}{3x}$.

But since any constant times a solution is a solution, we may choose for a second solution $\phi_2(x) = \frac{1}{x}$. Thus $x^2, \frac{1}{x}$ are solutions on $0 < x < \infty$.

Since $W(\phi_1, \phi_2)(x) = W(x^2, \frac{1}{x})(x) = -3 \neq 0$, $\{x^2, \frac{1}{x}\}$ is linearly independent.

Example 3.16. Consider the equation $x^2y'' - 7xy' + 15y = 0, \quad \phi_1(x) = x^3, \ \ x > 0$. Clearly $\phi_1(x) = x^3$ satisfies the given equation. Also

$$\begin{aligned}
\phi_2(x) = u(x)\phi_1(x) &= \phi_1(x)\int_{x_0}^{x} \frac{1}{[\phi_1(x)]^2}\ \exp\left[-\int_{x_0}^{x} a_1(x)\ dx\right] dx \\
&= x^3 \int \frac{1}{x^6}\ e^{\int \frac{7}{x}\ dx} dx \\
&= \frac{x^5}{2}
\end{aligned}$$

Since any constant times a solution is a solution we may choose a second solution $\phi_2(x) = x^5$. Also, since $W(\phi_1, \phi_2) = 2x^2 \neq 0$ as $x \neq 0$, $\{x^3, x^5\}$ is linearly independent.

3.6 The non-homogeneous equation

Theorem 3.17. *Let b be continuous on an interval I, and let $\phi_1, \cdots, \phi_n$ be n linearly independent solutions of $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0$ on I. Every solution ψ of $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x)$ can be written as*

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

where ψ_p is a particular solution of $L(y) = b(x)$ and $c_1, c_2, \cdots, c_n$ are constants. Every such ψ is a solution of $L(y) = b(x)$. A particular solution is given by

$$\psi_p = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{W_k(t)\ b(t)}{W(\phi_1, \phi_2, \cdots, \phi_n)(t)}\ dt.$$

Proof. Let $a_1(x), a_2(x), \cdots, a_n(x), b(x)$ be a continuous function on an interval I, and consider the equation

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x),$$

If ψ_p is a particular solution of $L(y) = b(x)$ and ψ is any other solution, then

$$L(\psi - \psi_p) = L(\psi) - L(\psi_p) = b - b = 0.$$

Thus $\psi - \psi_p$ is a solution of the homogeneous equation $L(y) = 0$, and this inplies that any solution ψ of $L(y) = b(x)$ can be written in the form

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

where ψ_p is a particular solution of $L(y) = b(x)$, the functions $\phi_1, \phi_2, \cdots, \phi_n$, are n linearly independent solutions of $L(y) = 0$, and $c_1, c_2, \cdots, c_n$ are constants.

To find a particular solution ψ_p, we proceed just as in the case $n = 2$, that is, we use the *variation of constants* method. We try to find n functions $u_1, u_2, \cdots u_n$ so that

$$\psi_p = u_1\phi_1 + u_2\phi_2 + \cdots + u_n\phi_n$$

is a solution. If

$$u_1'\phi_1 + u_2'\phi_2 + \cdots + u_n'\phi_n = 0,$$

then

$$\psi_p' = u_1\phi_1' + u_2\phi_2' \cdots + u_n\phi_n'$$

and if

$$u_1'\phi_1' + u_2'\phi_2' + \cdots + u_n'\phi_n' = 0,$$

then

$$\psi_p'' = u_1\phi_1'' + u_2\phi_2'' \cdots + u_n\phi_n''$$

Thus if $u_1', u_2', \cdots, u_n'$ satisfy

$$\begin{aligned} u_1'\phi_1 + u_2'\phi_2 + \cdots + u_n'\phi_n &= 0 \\ u_1'\phi_1' + u_2'\phi_2' + \cdots + u_n'\phi_n' &= 0 \\ &\vdots \\ u_1'\phi_1^{(n-2)} + u_2'\phi_2^{(n-2)} + \cdots + u_n'\phi_n^{(n-2)} &= 0 \\ u_1'\phi_1^{(n-1)} + u_2'\phi_2^{(n-1)} + \cdots + u_n'\phi_n^{(n-1)} &= b \end{aligned} \tag{3.20}$$

we see that

$$
\begin{aligned}
\psi_p &= u_1\phi_1 + u_2\phi_2 + \cdots + u_n\phi_n \\
\psi_p' &= u_1\phi_1' + u_2\phi_2' + \cdots + u_n\phi_n' \\
&\vdots \\
\psi_p^{(n-1)} &= u_1\phi_1^{(n-1)} + u_2\phi_2^{(n-1)} + \cdots + u_n\phi_n^{(n-1)} \\
\psi_p^{(n)} &= u_1\phi_1^{(n)} + u_2\phi_2^{(n)} + \cdots + u_n\phi_n^{(n)} + b
\end{aligned}
\tag{3.21}
$$

Hence $L(\psi_p) = u_1 L(\phi_1) + u_2 L(\phi_2) + \cdots + u_n L(\phi_n) + b = b,$

and indeed ψ_p, is a solution of $L(y) = b(x)$. The whole problem is noW reduced to solving the linear system (2.22) for $u_1', u_2', \cdots, u_n'$. The determinant of the coefficients is just $W(\phi_1, \phi_2, \cdots, \phi_n)$, which is never zero when $\phi_1, \cdots, \phi_n$ ae linearly independent solutions of $L(y) = 0$. Therefore there are unique functions $u_1', \cdots, u_n'$, satisfying (2.22). It is easy to see that solutions are given by

$$u_k'(x) = \frac{W_k(x)\ b(x)}{W(\phi_1, \phi_2, \cdots, \phi_n)(x)}, \qquad (k = 1, 2, \cdots, n)$$

where W_k is the determinant obtained from $W(\phi_1, \cdots, \phi_n)$ by replacing the $k-$th column (that is $\phi_k, \phi_k', \cdots, \phi_k^{(n-1)}$) by $0, 0, \cdots, 0, 1$.

If x_0 is any point in I we may take for u_k the function given by

$$u_k(x) = \int_{x_0}^{x} \frac{W_k(t)\ b(t)}{W(\phi_1, \phi_2, \cdots, \phi_n)(t)}\ dt, \qquad (k = 1, 2, \cdots, n)$$

The particular solution ψ_p now takes the form

$$\psi_p = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{W_k(t)\ b(t)}{W(\phi_1, \phi_2, \cdots, \phi_n)(t)}\ dt. \tag{3.22}$$

□

Example 3.18. Consider the equation $y'' - \dfrac{2}{x^2} y = x \quad (0 < x < \infty)$.

We have already see in section 3.5 that a basis for the solutions of the homogeneous equation is given by

$$\phi_1(x) = x^2, \qquad \phi_2(x) = x^{-1}.$$

A solution ψ_p of the non-homogeneous equation has the form

$$\psi_p = u_1 x^2 + u_2 x^{-1}$$

where u_1', u_2' satisfy

$$x^2u_1' + x^{-1}u_2' = 0 \text{ and } 2xu_1' - x^{-2}u_2' = x.$$

Now $W(\phi_1, \phi_2)(x) = -3$ and we find that $u_1'(x) = \dfrac{1}{3}, \quad u_2'(x) = \dfrac{-x^3}{3}$.

For u_1, u_2 we may take $u_1(x) = \dfrac{x}{3}, \quad u_2(x) = -\dfrac{x^4}{12}$,

and we see that $\psi_p(x) = (x^2)\left(\dfrac{x}{3}\right) - (x^{-1})\left(\dfrac{x^4}{12}\right) = \dfrac{x^3}{3} - \dfrac{x^3}{12} = \dfrac{x^3}{4}$.

Every solution ϕ of given equation has the form $\phi(x) = \dfrac{x^3}{4} + c_1x^2 + c_2x^{-1}$ where c_1, c_2 are constants.

Since we can always solve the non-homogeneous equation $L(y) = b(x)$ by using algebraic methods and an integration, we now concentrate our attention on methods for solving the homogeneous equation.

Exercise:

1. One solution of $x^2y'' - 2y = 0$ on $0 < x < \infty$ is $\phi_1(x) = x^2$. Find all solutions of $x^2y'' - 2y = 2x - 1$ on $0 < x < \infty$.
2. One solution of $x^2y'' - xy' + y = 0$ on $(x > 0)$, is $\phi_1(x) = x$. Find solution ψ of $x^2y'' - xy' + y = x^2$ satisfying $\psi(1) = 1, \ \psi(1) = 0$.

3.7 Homogeneous equations with analytic coefficients

If g is a function defined on an interval I containing a point x_0, we say that g is analytic at x_0 if g can be expanded in a power series about x_0 which has a positive radius of convergence. Thus g is analytic at x_0 if it can be represented in the form

$$g(x) = \sum_{k=0}^{\infty} c_k(x - x_0)^k, \tag{3.23}$$

where the c_k are constants, and the series converges for $|x - x_0| < r_0$, $r_0 > 0$. one of the important properties of a function g which has the form (3.23), where the series converges for $|x - x_0| < r_0$, is that all of its derivatives exist on $|x - x_0| < r_0$, and they may be computed by differentiating the series term by term. Thus, for example

$$g'(x) = \sum_{k=1}^{\infty} kc_k(x - x_0)^{k-1},$$

and

$$g''(x) = \sum_{k=2}^{\infty} k(k-1)c_k(x - x_0)^{k-2},$$

and the differentiated series converge on $|x - x_0| < r_0$ also. If the coefficients $a_1, \cdots, a_n$ of L are analyt1c at x_0 1t turns out that the solutions are also. In fact solutions can be computed by a formal algebraic process. We illustrate by considering the following example.

Example 3.19. Consider the equation $L(y) = y'' - xy = 0$. Here $a_1(x) = 0$ and $a_2(x) = -x$, and hence $a_1(x), a_2(x)$ are analytic for all real x_0. Now we try for the series solution. Consider the series

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k$$

Then

$$\phi'(x) = \sum_{k=1}^{\infty} kc_k x^{k-1}$$

$$\phi''(x) = \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k.$$

Also

$$x\phi(x) = \sum_{k=0}^{\infty} c_k x^{k+1} = \sum_{k=1}^{\infty} c_{k-1} x^k \quad .$$

Then

$$\begin{aligned} \phi''(x) - x\phi(x) &= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} c_{k-1} x^k \\ &= 2c_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - c_{k-1}\right] x^k. \end{aligned}$$

In order for ϕ to be a solution of $L(y) = 0$ we must have

$$\phi''(x) - x\phi(x) = 0.$$

That is $2c_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - c_{k-1}\right] x^k = 0$,

This is true only if all the coefficients of the powers of x are zero. Thus

$$2c_2 = 0, \quad (k+2)(k+1)c_{k+2} - c_{k-1} = 0, \quad (k = 1, 2, \cdots).$$

This gives an infinite set of equations, which can be solved for the c_k.

Thus, for $k = 1$, we have $3.2c_3 = c_0$. That is $c_3 = \dfrac{c_0}{3.2}$.

Then for $k = 2$ we have $c_4 = \dfrac{c_1}{4.3}$.

Continuing in this way we see that

$$c_5 = \frac{c_2}{5.4} = 0, \quad c_6 = \frac{c_3}{6.5} = \frac{c_0}{6.5.3.2}, \quad c_7 = \frac{c_4}{7.6} = \frac{c_1}{7.6.4.3}.$$

It can be shown by induction that

$$\begin{aligned} c_{3m} &= \frac{c_0}{2.3.5.6.\cdots(3m-1)3m}, && (m = 1, 2, 3, \cdots), \\ c_{3m+1} &= \frac{c_1}{3.4.6.7.\cdots 3m(3m+1)}, && (m = 1, 2, 3, \cdots), \\ c_{3m+2} &= 0 && (m = 0, 1, 2, \cdots). \end{aligned}$$

Thus all the constants are determined in terms of c_0 and c_1. Collecting together terms with c_0 and c_1 as a factor we have

$$\begin{aligned} \phi(x) &= \sum_{k=0}^{\infty} c_k x^k \\ &= c_0 + c_1 x + \sum_{k=2}^{\infty} c_k x^k \end{aligned}$$

$$\begin{aligned} &= c_0 + c_1 x + \sum_{m=1}^{\infty} \frac{c_0\ x^{3m}}{2.3.5.6.\cdots(3m-1)3m} + \sum_{m=1}^{\infty} \frac{c_1\ x^{3m+1}}{3.4.6.7.\cdots 3m(3m+1)} \\ &= c_0 \left[1 + \sum_{m=1}^{\infty} \frac{x^{3m}}{2.3.5.6.\cdots(3m-1)3m}\right] + c_1 \left[x + \sum_{m=1}^{\infty} \frac{x^{3m+1}}{3.4.6.7.\cdots 3m(3m+1)}\right] \end{aligned}$$

Let ϕ_1, ϕ_2 represent the two series in the brackets. Thus

$$\begin{aligned} \phi_1(x) &= 1 + \sum_{m=1}^{\infty} \frac{x^{3m}}{2.3.5.6.\cdots(3m-1)3m} \\ \phi_2(x) &= x + \sum_{m=1}^{\infty} \frac{x^{3m+1}}{3.4.6.7.\cdots 3m(3m+1)} \end{aligned} \tag{3.24}$$

Thus we have shown that ϕ satisfies $y'' - xy = 0$ for any two constants c_0, c_1. In

Particular, the choice $c_0 = 1, c_1 = 0$ shows that ϕ_1 satisfies this equation, and the choice $c_0 = 0, c_1 = 1$ shows that ϕ_2 also satisfies the equation. Next we have to check the convergence of the series $\phi_1(x), \phi_2(x)$. It can be checked easily by ratio test that both series converges for all finite x.

Let us consider the series for $\phi_1(x)$. Writing it as

$$\phi_1(x) = 1 + \sum_{m=1}^{\infty} d_m(x), \quad \text{where } d_m(x) = \sum_{m=1}^{\infty} \frac{x^{3m}}{2.3.5.6.\cdots(3m-1)3m}$$

we see that

$$\begin{aligned}
\frac{d_{m+1}}{d_m} &= \frac{x^{3m+3}}{2.3.5.6.\cdots(3m-1)(3m)(3m+2)(3m+3)} \times \frac{2.3.5.6.\cdots(3m-1)(3m)}{x^{3m}} \\
&= \frac{x^3}{(3m+2)(3m+3)} \\
\left|\frac{d_{m+1}}{d_m}\right| &= \left|\frac{x^3}{(3m+2)(3m+3)}\right| \\
&= \frac{|x|^3}{(3m+2)(3m+3)},
\end{aligned}$$

which tends to 0 as $m \to \infty$, provided only that $|x| < \infty$. Hence $\phi_1(x)$ is convergent. In the similar way, we can prove that $\phi_2(x)$ is convergent.

Next to check $\phi_1(x), \phi_2(x)$ are linearly independent solutions, it is clear from the series (3.24) defining ϕ_1 and ϕ_2 that

$$\phi_1(0) = 1, \quad \phi_2(0) = 0, \quad \phi_1'(0) = 0, \quad \phi_2'(0) = 1.$$

and therefore $W(\phi_1, \phi_2)(0) = \begin{vmatrix} \phi_1(0) & \phi_2(0) \\ \phi_1'(0) & \phi_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$

Hence ϕ_1, ϕ_2 are linearly independent.

Theorem 3.20. *(*Existence Theorem for Analytic Coefficients) Let x_0 be a real number and suppose that the coefficients $a_1, a_2, \cdots, a_n$ in

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$$

has convergent power series expansions in powers of $x - x_0$ on an interval $|x - x_0| < r_0, \ _0 > 0$. If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are any n constants, there exists a solution ϕ of the problem

$$L(y) = 0, \quad y(x_0) = \alpha_1, \quad , y^{(n-1)}(x_0) = \alpha_n,$$

with a power series expansion

$$\phi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

convergent for $|x-x_0|<r_0$. We have $k!\ c_k=\alpha_{k+1}, \quad (k=0,1,2,\cdots,n-1)$, and c_k for $k\geq n$ may be computed in terms of $c_0,c_1,\cdots,c_{n-1}$ by substituting the above series into $L(y)=0$.

Exercise:
1. Find two linearly independent power series solutions(in powers of x) of the following equations.

(a) $y''+y=0$ (b) $y''-xy'+y=0$
(c) $y''-x^2y=0$ (d) $y''+x^3y'+x^2y=0$

3.8 The Legendre equation

Some of the important differential equations met in physical problems are second order linear equations with analytic coefficients. One of these is the Legendre equation

$$L(y)=(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0 \tag{3.25}$$

where α is a constant. If we write this equation as

$$y''-\frac{2x}{1-x^2}y'+\frac{\alpha(\alpha+1)}{1-x^2}y=0,$$

we see that the functions a_1,a_2 given by

$$a_1(x)=\frac{-2x}{1-x^2}, \qquad a_2(x)=\frac{\alpha(\alpha+1)}{1-x^2}$$

are analytic at $x=0$. Indeed,

$$\frac{1}{1-x^2}=1+x^2+x^4+\cdots=\sum_{k=0}^{\infty}x^{2k},$$

and this series converges for $|x|<1$. Thus a_1 and a_2 have hte series expansions

$$a_1(x)=\textstyle\sum_{k=0}^{\infty}(-2)x^{2k+1}, \qquad a_2(x)=\sum_{k=0}^{\infty}\alpha(\alpha+1)x^{2k},$$

which converge for $|x|<1$. From Theorem 3.20 it follows that the solutions of $L(y)=0$ on $|x|<1$ have convergent power series expansions there. We proceed to find a basis for these solutions.

Let ϕ be any solution of the Legendre equation on $|x|<1$, and suppose

$$\phi(x)=c_0+c_1x+c_2x^2+\cdots=\sum_{k=0}^{\infty}c_kx^k. \tag{3.26}$$

We have $\phi'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{k=0}^{\infty} k c_k x^{k-1}.$

$$-2x\phi'(x) = \sum_{k=0}^{\infty} -2k c_k x^k. \tag{3.27}$$

$$\phi''(x) = 2c_2 x + 3.2c_3 x + \cdots = \sum_{k=0}^{\infty} k(k-1)c_k x^{k-2},$$

$$-x^2\phi'(x) = \sum_{k=0}^{\infty} k(k-1)c_k x^k. \tag{3.28}$$

Note that $\phi''(x)$ may also be written as

$$\phi''(x) = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k, \tag{3.29}$$

From (3.26)-(3.29) we obtain

$$L(\phi)(x) = (1-x^2)\phi''(x) - 2x\phi'(x) + \alpha(\alpha+1)\phi(x)$$

$$\begin{aligned} L(\phi)(x) &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)c_k x^k - \sum_{k=0}^{\infty} 2k c_k x^k \\ &\quad + \alpha(\alpha+1)\sum_{k=0}^{\infty} c_k x^k \\ &= \sum_{k=0}^{\infty}\left[(k+2)(k+1)c_{k+2} - k(k-1)c_k - 2kc_k + \alpha(\alpha+1)c_k\right]x^k \\ &= \sum_{k=0}^{\infty}\left[(k+2)(k+1)c_{k+2} + (\alpha+k+1)(\alpha-k)c_k\right]x^k, \end{aligned}$$

since $-k(k-1) - 2k + \alpha(\alpha+1) = -k(k+1) + \alpha(\alpha+1) = -k(k+1) + \alpha(\alpha+1) + \alpha k - \alpha k = -k(\alpha+k+1) + \alpha(\alpha+k+1) = (\alpha+k+1)(\alpha-k)$.
For ϕ to satisfy $L(\phi) = 0$ we must have all the coefficients of the powers of x equal to zero. Hence

$$(k+2)(k+1)c_{k+2} + (\alpha+k+1)(\alpha-k)c_k = 0, \quad (k = 0, 1, 2, \cdots) \tag{3.30}$$

This is the recursion relation which gives c_{k+2} in terms of c_k. For $k = 0$ we have

$$c_2 = -\frac{(\alpha+1)\alpha}{2}\ c_0,$$

For $k = 1$ we get

$$c_3 = -\frac{(\alpha+2)(\alpha-1)}{3.2}\ c_1,$$

Similarly, for $k = 2, 3$ we obtain

$$c_4 = -\frac{(\alpha+3)(\alpha-2)}{4.3}\ c_2 = \frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{4.3.2}\ c_0$$

$$c_5 = -\frac{(\alpha+4)(\alpha-3)}{5.4}\ c_3 = \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{5.4.3.2}\ c_1$$

The pattern now becomes clear, and it follows by induction that for $m = 1, 2, \cdots$,

$$c_{2m} = (-1)^m \frac{\alpha(\alpha+1)(\alpha+3)\cdots(\alpha+2m-1)(\alpha-2)(\alpha-4)\cdots(\alpha-2m+2)}{(2m)\ !}\ c_0,$$

$$c_5 = (-1)^m \frac{(\alpha+2)(\alpha+4)\cdots(\alpha+2m)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)\ !}\ c_1$$

All coefficients are determined in terms of $c_0,\ c_1$, and we must have

$$\phi(x) = c_0\ \phi_1(x) + c_1\ \phi_2(x)$$

where

$$\phi_1(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{\alpha(\alpha+1)(\alpha+3)\cdots(\alpha+2m-1)(\alpha-2)(\alpha-4)\cdots(\alpha-2m+2)}{(2m)\ !}$$

$$\phi_2(x) = x + (-1)^m \frac{(\alpha+2)(\alpha+4)\cdots(\alpha+2m)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)\ !}$$

Both $\phi_1,\ \phi_2$ are solutions of the Legendre equation, those corresponding to the choices

$$c_0 = 1,\quad c_1 = 0 \qquad \text{and} \qquad c_0 = 0,\quad c_1 = 1,$$

respectively. They form a basis for the solutions, since

$$\phi_1(0) = 1,\quad \phi_2(0) = 0\ ;\quad \phi_1'(0) = 0,\quad \phi_2'(0) = 1.$$

We notice that if α is a non-negative even integer $n = 2m$, $\quad (m = 0, 1, 2, \cdots)$,

then ϕ_1 has only a finite number of non-zero terms. Indeed, in this case ϕ_1 is a polynomial of degree n containing only even powers of x. For example,

$$\begin{aligned}\phi_1(x) &= 1, && (\alpha = 0),\\ \phi_1(x) &= 1 - 3x^2, && (\alpha = 2),\\ \phi_1(x) &= 1 - 10x^2 + \frac{35}{3}x^4, && (\alpha = 4).\end{aligned}$$

The solution ϕ_2 is not a polynomial in this case since none of the coefficients in the series of $\phi_2(x)$ vanish.

A similar situation occurs when α is a positive odd integer n. Then ϕ_2 is a polynomial of degree n having only odd powers of x, and ϕ_1 is not a polynomial. For example,

$$\begin{aligned}\phi_2(x) &= x, && (\alpha = 1),\\ \phi_2(x) &= x - \frac{5}{3}x^3, && (\alpha = 3),\\ \phi_2(x) &= x - \frac{14}{3}x^3 + \frac{21}{5}x^5, && (\alpha = 5).\end{aligned}$$

We consider in more detail these polynomial solutions when $\alpha = n$, non-negative integer. The polynomial solution P_n, of degree n of

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \tag{3.31}$$

Sstisfying $P_n(1) = 1$ is called the $n-$th *Legendre polynomial.* In order to justify this definition we must show that there is just one such solution for each non-negative integer n.

Let ϕ be the polynomial of degree n defined by

$$\phi(x) = \frac{d^n}{dx^n}(x^2 - 1)^n.$$

This ϕ satisfies the Legendre equation (3.31). Indeed, let

$$u(x) = (x^2 - 1)^n.$$

Then we obtain by differentiating

$$\begin{aligned}u'(x) &= n(x^2 - 1)^{n-1}(2x)\\ &= 2nx(x^2 - 1)^n(x^2 - 1)^{-1}\\ (x^2 - 1)u'(x) &= 2nx(x^2 - 1)^n\\ (x^2 - 1)u'(x) &= 2nx\, u(x)\end{aligned}$$

Thus $(x^2-1)u'(x)-2nx\ u(x)=0$. Differentiating this expression $n+1$ times yields

$$(x^2-1)u^{(n+2)}+2x(n+1)\ u^{(n+1)}+(n+1)n\ u^{(n)}-2nxu^{(n+1)}-2n(n+1)\ u^{(n)}=0.$$

Note: $(fg)^{(n)}=fg^{(n)}+\binom{n}{1}f'g^{(n-1)}+\binom{n}{2}f''g^{(n-2)}+\cdots+\binom{n}{1}f^{(n-1)}g'+f^{(n)}g.$

Putting $\phi=\dfrac{d^n}{dx^n}(x^2-1)^n=u^{(n)}$ we obtain

$$\begin{aligned}(x^2-1)\phi''+2(n+1)x\phi'+n(n+1)\phi-2nx\phi'-2n(n+1)\phi &= 0\\ (x^2-1)\phi''+2x\phi'-n(n+1)\phi &= 0\\ (1-x^2)\phi''-2x\phi'+n(n+1)\phi &= 0\end{aligned}$$

We have shown that ϕ satisfies (3.31). Thus ϕ is a solution of (3.31). This polynomial ϕ satisfies

$$\phi(1)=2^n\ n!.$$

This can be seen by noting that

$$\begin{aligned}\phi(x) &= \frac{d^n}{dx^n}(x^2-1)^n\\ &= \frac{d^n}{dx^n}(x^2-1)^n\\ &= \left[(x^2-1)^n\right]^{(n)}\\ &= [(x-1)^n(x+1)^n]^{(n)}\\ &= [(x-1)^n]^{(n)}\ (x+1)^n+\text{ terms with }(x-1)\text{ as a factor}\\ &= n\ !(x+1)^n+\text{ terms with }(x-1)\text{ as a factor.}\end{aligned}$$

Hence $\phi(1)=n\ !2^n$ as stated.

It is clear that the function P_n given by

$$P_n(x)=\frac{1}{2^n n\ !}\ \frac{d^n}{dx^n}(x^2-1)^n$$

is the solution of (3.31) and it is the Legendre polynomial provided that $P_n(1)=1$. This $P_n(x)$ is known as Rodrigues formula.

Note:

$$
\begin{aligned}
P_0(x) &= \frac{1}{2^0 0\ !}\ (x^2-1)^0 = 1 \\
P_1(x) &= \frac{1}{2^1 1\ !}\ \frac{d}{dx}(x^2-1) = \frac{2x}{2} = x \\
P_2(x) &= \frac{1}{2^2 2\ !}\ \frac{d^2}{dx^2}(x^2-1)^2 = \frac{3}{2}x^2 - \frac{1}{2} \\
P_3(x) &= \frac{1}{2^3 3\ !}\ \frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{2}(5x^3-3x)
\end{aligned}
$$

Properties of Legendre Polynomial

Generating function: The function $G(t,x)$ given by

$$G(t,x) = \frac{1}{\sqrt{1-2xt+t^2}}$$

is called generating function of Legendre polynomial.

Note: Expanding $\frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)^{-1/2}$ by binomial expansion we get the relation

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

Recurrence Relations:

1. $\mathbf{(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)}$

   Proof: We know that $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$

   Differentiating with respect to t we have

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1-2xt+t^2}}\right) = \frac{d}{dt}\left(\sum_{n=0}^{\infty} P_n(x)t^n\right).$$

$$-\frac{1}{2}\frac{-2(x-t)}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

$$\frac{(x-t)}{(1-2xt+t^2)}\ \frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

$$\frac{(x-t)}{(1-2xt+t^2)}\ \sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

$$(x-t)\ \sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)\ \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

$$\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n-1} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1}$$

$$\begin{aligned}\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=0}^{\infty} P_{n-1}(x)t^n = \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - \sum_{n=0}^{\infty} 2nxP_n(x)t^n \\ + \sum_{n=0}^{\infty}(n-1)P_{n-1}(x)t^n\end{aligned}$$

$$\sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n = \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n + \sum_{n=0}^{\infty} nP_{n-1}(x)t^n$$

Equating the coefficient of t^n we have

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

2. $\mathbf{(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)}$

Proof: We know that

$$\begin{aligned}
P_k(x) &= \frac{1}{2^k k!}\,\frac{d^k}{dx^k}(x^2-1)^k \\
P'_k(x) &= \frac{d}{dx}\left(\frac{1}{2^k k!}\,\frac{d^k}{dx^k}(x^2-1)^k\right) \\
&= \frac{1}{2^k k!}\,\frac{d^k}{dx^k}\left(\frac{d}{dx}\left((x^2-1)^k\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^k k!}\,\frac{d^k}{dx^k}\left(k(x^2-1)^{k-1}(2x)\right) \\
&= \frac{2k}{2^k k!}\,\frac{d^k}{dx^k}\left(x(x^2-1)^{k-1}\right) \\
&= \frac{1}{2^{k-1}(k-1)!}\,\frac{d^{k-1}}{dx^{k-1}}\left(\frac{d}{dx}\left(x(x^2-1)^{k-1}\right)\right) \\
&= \frac{1}{2^{k-1}(k-1)!}\,\frac{d^{k-1}}{dx^{k-1}}\left((k-1)x(x^2-1)^{k-2}(2x) + (x^2-1)^{k-1}\right) \\
P'_k(x) &= \frac{1}{2^{k-1}(k-1)!}\,\frac{d^{k-1}}{dx^{k-1}}\left((x^2-1)^{k-2}\left((2k-1)x^2-1\right)\right)
\end{aligned}$$

Thus for $k = n+1$ we have $P'_{n+1}(x) = \frac{1}{2^n n!}\,\frac{d^n}{dx^n}\left((x^2-1)^{n-1}\left((2n+1)x^2-1\right)\right)$,

From Rodrigues formula at $n-1$, we have

$$P_{n-1}(x) = \frac{1}{2^{n-1}(n-1)!}\,\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}$$

$$P'_{n-1}(x) = \frac{d}{dx}\left(\frac{1}{2^{n-1}(n-1)!}\,\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right)$$

$$P'_{n-1}(x) = \frac{2n}{2^n n!}\left(\frac{d^n}{dx^n}(x^2-1)^{n-1}\right)$$

As a consequence we have

$$\begin{aligned}
P'_{n+1}(x) - P'_{n-1}(x) &= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^{n-1}\left((2n+1)x^2-1\right)\right) \\
&\quad -\frac{2n}{2^n n!}\left(\frac{d^n}{dx^n}(x^2-1)^{n-1}\right) \\
&= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[((2n+1)x^2-1)(x^2-1)^{n-1} - 2n(x^2-1)^{n-1}\right] \\
&= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^{n-1}(2nx^2+x^2-1-2n)\right] \\
&= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^{n-1}(2n(x^2-1)+(x^2-1))\right] \\
&= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^{n-1}((2n+1)(x^2-1))\right] \\
&= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^{n}(2n+1)\right] \\
&= \frac{2n+1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^{n}\right] \\
P'_{n+1}(x) - P'_{n-1}(x) &= (2n+1)P_n(x)
\end{aligned}$$

Hence $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

3. $\mathbf{xP'_n(x) - P'_{n-1}(x) = nP_n(x)}$

   Proof: We know that

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \qquad (3.32)$$

Differentiating (3.32) with respect to t, we have

$$-\tfrac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

$$(1-2xt+t^2)^{-3/2}(x-t) = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} \qquad (3.33)$$

Differentiating (3.32) with respect to x, we have

$$-\tfrac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} P'_n(x)t^n$$

$$(1-2xt+t^2)^{-3/2}(t) = \sum_{n=0}^{\infty} P'_n(x)t^n \tag{3.34}$$

Dividing (3.33) by (3.34),

$$\begin{aligned}
\frac{(x-t)(1-2xt+t^2)^{-3/2}}{t(1-2xt+t^2)^{-3/2}} &= \frac{\sum_n nP_n(x)t^{n-1}}{\sum_n P'_n(x)t^n} \\
\frac{(x-t)}{t} &= \frac{\sum_n nP_n(x)t^{n-1}}{\sum_n P'_n(x)t^n} \\
(x-t)\sum_n P'_n(x)t^n &= t\sum_n nP_n(x)t^{n-1} \\
\sum_n xP'_n(x)t^n - \sum_n P'_n(x)t^{n+1} &= \sum_n nP_n(x)t^n \\
\sum_n xP'_n(x)t^n - \sum_n P'_{n-1}(x)t^n &= \sum_n nP_n(x)t^n
\end{aligned}$$

Equating coefficient of t^n, we have

$$xP'_n(x) - P'_{n-1}(x) = nP_n(x)$$

4. $\mathbf{P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x)}$

Proof: We know that

$$P'n+1(x) - P'n-1(x) = (2n+1)P_n(x) \tag{3.35}$$

$$xP'_n(x) - P'_{n-1}(x) = nP_n(x) \tag{3.36}$$

$$\begin{aligned}
P'n+1(x) - xP'_n(x) &= P'_{n+1}(x) - P'_{n-1}(x) - nP_n(x), \quad \text{using (3.36)} \\
&= (2n+1)P_n(x) - nP_n(x), \quad \text{using (3.35)} \\
&= (n+1)P_n(x)
\end{aligned}$$

Hence $P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x)$

Orthogonal property:

This is the most important property of Legendre polynomial.

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases} \tag{3.37}$$

Proof:

Let $f(x)$ be any function with atleast n continuous derivatives on the interval $-1 \leq x \leq 1$. Consider the integral

$$I = \int_{-1}^{1} f(x)P_n(x)dx$$

We know that $P_n(x) = \dfrac{1}{2^n n\,!}\ \dfrac{d^n}{dx^n}(x^2-1)^n$

$$\begin{aligned} I &= \int_{-1}^{1} f(x)\frac{1}{2^n n\,!}\ \frac{d^n}{dx^n}(x^2-1)^n dx \\ &= \frac{1}{2^n n\,!}\int_{-1}^{1} f(x)\frac{d^n}{dx^n}(x^2-1)^n dx \end{aligned}$$

Applying integration by parts (i.e., $\int udv = uv - \int vdu$), by taking $u = f(x),\ dv = \dfrac{d^n}{dx^n}(x^2-1)^n$ we have

$$I = \frac{1}{2^n n\,!}\left[f(x)\ \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1} - \frac{1}{2^n n\,!}\int_{-1}^{1} f'(x)\ \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$$

That is $I = \dfrac{-1}{2^n n\,!}\displaystyle\int_{-1}^{1} f'(x)\ \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$, since the first term is zero after applying the limit.

By continuing the integration by parts n times, we obtain

$$I = \frac{(-1)^n}{2^n n\,!}\int_{-1}^{1} f^{(n)}(x)\ (x^2-1)^n dx$$

If $f(x) = P_m(x)$ with $m < n$, then $f^{(n)}(x) = 0$ and so $I = 0$ which proves the first part of (3.37). Now put $f(x) = P_n(x)$.

$$\begin{aligned} I &= \int_{-1}^{1} f(x)P_n(x)dx \\ &= \int_{-1}^{1} \frac{1}{2^n n\,!}\,\frac{d^n}{dx^n}(x^2-1)^n\,\frac{1}{2^n n\,!}\,\frac{d^n}{dx^n}(x^2-1)^n\;dx \end{aligned}$$

Applying integration by parts by taking $u = \frac{d^n}{dx^n}(x^2-1)^n$ and $dv = \frac{d^n}{dx^n}(x^2-1)^n$, we have

$$I = \frac{-1}{(2^n n\,!)^2}\int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\;dx.$$

By continuing the integration by parts n times, we obtain

$$\begin{aligned} I &= \frac{(-1)^n}{(2^n n\,!)^2}\int_{-1}^{1}(x^2-1)^n\,\frac{d^{2n}}{dx^{2n}}(x^2-1)^n\;dx. \\ &= \frac{(-1)^n(2n)!}{(2^n n\,!)^2}\int_{-1}^{1}(x^2-1)^n dx, \quad \text{since} \ \frac{d^{2n}}{dx^{2n}}(x^2-1)^n = (2n)! \\ &= \frac{(2n)!}{(2^n n\,!)^2}\int_{-1}^{1}(1-x^2)^n dx = \frac{2(2n)!}{(2^n n\,!)^2}\int_{0}^{1}(1-x^2)^n dx \end{aligned}$$

By change of variable, $x = \sin\theta$ we have $dx = \cos\theta\; d\theta$.

Also the limit when $x = 1, \theta = 0;\ x = 1, \theta = \frac{\pi}{2}$. Then

$$\begin{aligned}
I &= \frac{2(2n)!}{(2^n n\,!)^2} \int_0^1 (1-x^2)^n dx \\
&= \frac{2(2n)!}{(2^n n\,!)^2} \int_0^{\pi/2} (1-\sin^2 x)^n \cos\theta \ d\theta \\
&= \frac{2(2n)!}{(2^n n\,!)^2} \int_0^{\pi/2} \cos^{2n}\theta \cos\theta \ d\theta \\
&= \frac{2(2n)!}{(2^n n\,!)^2} \int_0^{\pi/2} \cos^{2n+1}\theta \ d\theta \\
&= \frac{2(2n)!}{(2^n n\,!)^2} \frac{(2n)(2n-2)\cdots 4.2}{(2n+1)(2n-1)\cdots 5.3}, \\
& \quad \text{Using the result } \int_0^{\pi/2} \cos^{2n}\theta \ d\theta = \frac{(2n-1)(2n-3)\cdots 3.1}{(2n)(2n-2)\cdots 4.2} \\
&= \frac{2(2n)!}{2^{2n}(n\,!)^2} \frac{(2^n(n)(n-1)\cdots 2.1)((2n)(2n-2)\cdots 4.2)}{(2n+1)(2n)(2n-1)\cdots 5.4.3.2.1} \\
&= \frac{2(2n)!}{2^{2n}(n\,!)^2} \frac{(2^n(n)(n-1)\cdots 2.1)(2^n(n)(n-1)\cdots 2.1)}{(2n+1)!} \\
&= \frac{2(2n)!}{2^{2n}(n\,!)^2} \frac{(2^n n!)(2^n n!)}{(2n+1)!} \\
I &= \frac{2}{2n+1}
\end{aligned}$$

Hence the proof.

Chapter 4

Linear equations with regular singular points

4.1 Introduction

We consider the linear equation with variable coefficients

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0. \tag{4.1}$$

We shall assume that the coefficients $a_0, a_1, \cdots, a_n$ are analytic at some point x_0 and we shall be interested in an important case when $a_0(x_0) = 0$. A point x_0 such that $a_0(x_0) = 0$ is called a *singular point* of the equation (4.1).

We say that x_0 is a *regular singular point* for (4.1) if the equation can be written in the form

$$(x - x_0)^n y^{(n)} + b_1(x)(x - x_0)^{n-1} y^{(n-1)} + \cdots + b_n(x)y = 0 \tag{4.2}$$

near x_0 where the functions $b_1, \cdots, b_n$ are analytic at x_0. If the function $b_1, \cdots, b_n$ can be written in the form

$b_n(x) = (x - x_0)^k \beta_k(x), \qquad (k = 1, 2, \cdots, n),$

where $\beta_1, \beta_2, \cdots, \beta_n$ are analytic at x_0, we see that (4.2) becomes

$$y^{(n)} + \beta_1(x)y^{(n-1)} + \cdots + \beta_n(x)y = 0 \tag{4.3}$$

upon dividing out $(x-x_0)^n$. Thus (4.2) is a generalization of the equation with analytic coefficients. An equation of the form

$$c_0(x)(x-x_0)^n y^{(n)} + c_1(x)(x-x_0)^{n-1} y^{(n-1)} + \cdots + c_n(x)y = 0$$

has a regular singular point at x_0 if $c_0, c_1, \cdots, c_n$ are analytic at x_0, and $c_0(x_0) \neq 0$. This is because we may divide by $c_0(x)$, for x near x_0, to obtain an equation of the form (4.2) with $b_k(x) = c_k(x)/c_0(x)$, and it can be shown that these b_k are analytic at x_0.

We first consider the simplest case of an equation, not of the type (4.3), having a regular singular point. This is the Euler equation, which is the case of (4.2) with $b_1, \cdots, b_n$ all constants. Next we investigate the general equation of the second order with a regular singular point, and indicate how solutions may be obtained near the singular point. For $x > x_0$ such solutions ϕ turn out to be of the form

$$\phi(x) = (x-x_0)^r \sigma(x) + (x-x_0)^s \rho(x) \log(x-x_0),$$

where r, s are constants and σ, ρ are analytic at x_0.

Consider the equation

$$x^2 y'' - y' - \frac{3}{4} y = 0 \tag{4.4}$$

. The origin $x_0 = 0$ is a singular point, but not a regular singular point since the coefficient -1 of y' is not of the form $x b_1(x)$, where b_1 is analytic at 0. We may solve this equation by a series

$$\sum_{k=0}^{\infty} c_k x^k, \tag{4.5}$$

where the coefficients c_k satisfy the recursion formula

$$(k+1)\ c_{k+1} = \left(k^2 - k - \frac{3}{4}\right) c_k, \qquad (k = 0, 1, 2, \cdots) \tag{4.6}$$

If $c_0 \neq 0$, the ratio test applied to (4.5),(4.6), shows that

$$\left|\frac{c_{k+1}x^{k+1}}{c_k x^k}\right| = \left|\frac{k^2 - k - \frac{3}{4}}{k+1}\right| |x| \to \infty, \text{ as } k \to \infty,$$

provided $|x| \neq 0$. Thus the series (4.5) will only converge for $x = 0$.

4.2 The Euler equation

The simplest example of a second order equation having a regular singular point at the origin is the Euler equation.

Theorem 4.1. *Consider ther second order Euler equation*

$$L(y) = x^2y'' + axy' + by = 0,$$

where a, b are constants, and the polynomial q given by

$$q(r) = r(r-1) + ar + b$$

.

A basis for the solutions of the Euler equation on any interval not containing $x = 0$ is given by

$$\phi_1(x) = |x|^{r_1}, \quad \phi_2(x) = |x|^{r_2},$$

in case r_1, r_2 are distinct roots of q and by

$$\phi_1(x) = |x|^{r_1}, \quad \phi_2(x) = |x|^{r_1} \log |x|,$$

if r_1 is a root of multiplicity two.

Proof. Consider the equation

$$L(y) = x^2y'' + axy' + by = 0, \tag{4.7}$$

where a, b are constants. We first consider this equation for $x > 0$, and observe that the coeficient of $y^{(k)}$ in $L(y)$ is a constant times x^k. If r is any constant, x^r has the property that its $k-$th derivative times x^k is a constant times x^r. For example

$$x(x^r)' = rx^r, \qquad x^2(x^r)'' = r(r-1)x^r.$$

This suggests trying for a solution of $L(y) = 0$ a power of z. We find that

$$L(x^r) = [r(r-1) + ar + b]x^r.$$

If q is the polynomial defined by

$$q(r) = r(r-1) + ar + b,$$

we may write

$$\begin{aligned} L(x^r) &= x^2(x^r)'' + ax(x^r)' + b(x^r) \\ &= r(r-1)x^r + arx^r + bx^r \\ &= (r(r-1) + ar + b)x^r \\ L(x^r) &= q(r)x^r \end{aligned} \tag{4.8}$$

and it is clear that if r_1 is a root of $q(r)$ then $q(r_1) = 0$. Therefore

$$L(x_1^r) = q(r_1)x^r = 0.$$

Thus the functionϕ_1 given by $\phi_1(x) = x^{r_1}$ is a solution of (4.7) for $x > 0$.

Case 1: If r_2 is the other root of q, and $r_2 \neq r_1$, then we obtain another solution ϕ_2 given by $\phi_2(x) = x^{r_2}$.

claim: ϕ_1 and ϕ_2 are linearly independent in the case $r_1 \neq r_2$.

Suppose c_1, c_2 are constants such that

$$c_1 x^{r_1} + c_2 x^{r_2} = 0, \qquad (x > 0)$$

,

then

$$c_1 + c_2 x^{r_2 - r_1} = 0, \qquad (x > 0). \tag{4.9}$$

Differentiating we see that $c_2(r_2 - r_1)c^{r_2 - r_1 - 1} = 0$, which implies $c_2 = 0$, since $r_2 - r_1 \neq 0$ and $x \neq 0$. From (4.9) we obtain $c_1 = 0$ also. Hence ϕ_1, ϕ_2 are linearly independent.

Case 2: The roots r_1, r_2 of q are equal then $q(r_1) = 0, q'(r_1) = 0$, and this suggests differentiating (4.8) with respect to r. Indeed

$$\begin{aligned} \frac{\partial}{\partial r} L(x^r) &= L\left(\frac{\partial}{\partial r} x^r\right) \\ &= L\left(\frac{\partial}{\partial r} e^{\log x^r}\right) \\ &= L\left(\frac{\partial}{\partial r} e^{r \log x}\right) \\ &= L\left(e^{r \log x} \log x\right) \\ &= L(x^r \log x) \end{aligned}$$

$L(x^r \log x) = x^2(x^r \log x)'' + ax(x^r \log x)' + b(x^r \log x) = [q(r) \log x + q'(r)]x^r$

Since r_1 is a equal root of q, $q(r_1) = 0$ and $q'(r_1) = 0$. Then $L(x^{r_1} \log x) = 0$. Therefore $\phi_2(x) = x^{r_1} \log x$ is a second solutions of (4.7) in this case.

claim: ϕ_1 and ϕ_2 are linearly independent in the case $r_1 = r_2$.

Suppose c_1, c_2 are constants such that $c_1\phi_1 + c_2\phi_2 = 0$. That is

$$c_1 x^{r_1} + c_2 x^{r_2} \log x = 0, \qquad (x > 0)$$

,

then

$$c_1 + c_2 \log x = 0, \qquad (x > 0). \tag{4.10}$$

Differentiating we obtain $\frac{c_2}{x} = 0$ for $(x > 0)$, which implies $c_2 = 0$, since $r_2 - r_1 \neq 0$ and $x \neq 0$. From (4.10) we obtain $c_1 = 0$ also. Hence ϕ_1, ϕ_2 are linearly independent.

In either case the solutions ϕ_1 and ϕ_2 are linearly independent for $x > 0$.

We define x^r for r complex by $x^r = e^{r \log x}, \quad (x > 0)$.

Then we have $(x^r)' = (e^{r \log x})' = e^{r \log x}(r \log x)' = \dfrac{r}{x} e^{r \log x} = \dfrac{r}{x} x^r = r x^{r-1}$,

and $\dfrac{\partial}{\partial r}(x^r) = \dfrac{\partial}{\partial r}(e^{r \log x}) = \log x (e^{r \log x}) = x^r \log x$,

which are the formulas we used in the calculations. Solutions for (4.7) can be found for $x < 0$ also. In this case consider $(-x)^r$, where r is a constant. Then we have for $x < 0$,

$$[(-x)^r]' = -r(-x)^{r-1}, \quad [(-x)^r]'' = r(r-1)(-x)^{r-2},$$

and hence $x\,[(-x)^r]' = -r(-x)^r, \quad x^2\,[(-x)^r]'' = r(r-1)(-x)^r$.

Thus $L((-x)^r) = q(r)(-x)^r, \quad (x < 0)$.

Also $\dfrac{\partial}{\partial r}[(-x)^r] = (-x)^r \log(-x), \quad (x < 0)$.

Therefore we see that if the roots r_1, r_2 of q are distinct, then two independent solutions ϕ_1, ϕ_2 of (4.7) for $x < 0$ are given by

$$\phi_1(x) = (-x)^{r_1}, \quad \phi_2(x) = (-x)^{r_2}, \qquad (x < 0)$$

and if $r_1 = r_2$, then two solutions are given by

$$\phi_1(x) = (-x)^{r_1}, \quad \phi_2(x) = (-x)^{r_1} \log(-x), \qquad (x < 0)$$

These are just the formulas for the solutions obtained for $x > 0$, with x replaced by $-x$ everywhere. Since $|x| = x$ for $x > 0$, and $|x| = -x$ for $x < 0$, we can write the solutions for any $x \neq 0$ in the following way:

$$\phi_1(x) = |x|^{r_1}, \quad \phi_2(x) = |x|^{r_2}, \quad (x \neq 0),$$

in case $r_1 \neq r_2$ and

$$\phi_1(x) = |x|^{r_1}, \quad \phi_2(x) = |x|^{r_1} \log|x|, \quad (x \neq 0),$$

in case $r_1 = r_2$. □

Example 4.2. *Consider the equation $x^2y'' + xy' + y = 0$ for $x \neq 0$.*

The polynomial q is given by

$$q(r) = r(r-1) + r + 1 = r^2 + 1$$

and its roots are $r_1 = i$ and $r_2 = -i$. Thus a basis for the solutions is given by

$$\phi_1(x) = |x|^i, \quad \phi_2(x) = |x|^{-i}, \quad (x \neq 0),$$

where $|x|^i = e^{i \log|x|}$. Note that in this case another basis ψ_1, ψ_2 is given by

$$\psi_1(x) = \cos(\log|x|), \quad \psi_2(x) = \sin(\log|x|), \quad (x \neq 0).$$

Remark 4.3. The above theorem can be extended to n^{th} order equation

$$L(y) = x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y = 0 \tag{4.11}$$

where $a_1, a_2, \cdots, a_n$ are constants. Then for any constant r we have $L(|x|^r) = q(r)|x|^r$ where $q(r) = r(r-1)\cdots(r-n+1) + a_1 r(r-1)\cdots(r-n+2) + \cdots + a_n$.

This polynomial is called the indicial polynomial for the Euler equation (4.11). If r_1 is a root of q of multiplicity m, then $q(r_1) = 0,\ q'(r_1) = 0,\ \cdots,\ q^{(m_1-1)}(r_1) = 0$, and we see that

$$|x|^{r_1}, \ |x|^{r_1} \log|x|, \ \cdots, \ |x|^{r_1} \log^{m_1-1}|x|$$

are solutions of $L(y) = 0$. Repeating this process for each root of q we obtain the following result.

Theorem 4.4. *Let $r_1, r_2, \cdots, r_s$ be the distinct roots of the indicial polynomial q for $L(y) = x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y = 0$ and suppose r_i has multiplicity m_i. Then the n functions*

$$\begin{array}{c}
|x|^{r_1}, \ |x|^{r_1} \log|x|, \ \cdots, \ |x|^{r_1} \log^{m_1-1}|x|; \\
|x|^{r_2}, \ |x|^{r_2} \log|x|, \ \cdots, \ |x|^{r_2} \log^{m_2-1}|x|; \\
\vdots \\
|x|^{r_s}, \ |x|^{r_s} \log|x|, \ \cdots, \ |x|^{r_s} \log^{m_s-1}|x|
\end{array}$$

form a basis for the solutions of the $n-^{th}$ order Euler equation on any interval not *containing* $x = 0$.

Exercise:

1. Find all solutions of the following equations for $x > 0$:

(a) $x^2y'' + 2xy' - 6y = 0$ (b) $2x^2y'' + xy' - y = 0$

(c) $x^2y'' + xy' - 4y = x$ (d) $x^2y'' - 5xy' + 9y = x^3$

4.3 Second order equations with regular singular points-an example

A second order equation with a regular singular point at x_0 has the form

$$(x - x_0)^2y'' + a(x)(x - x_0)y' + b(x)y = 0, \tag{4.12}$$

where a, b are analytic at x_0. Thus a, b have power series expansions

$$a(x) = \sum_{k=0}^{\infty} \alpha_k(x - x_0)^k, \qquad b(x) = \sum_{k=0}^{\infty} \beta_k(x - x_0)^k,$$

which are convergent on some interval $|x - x_0| < r_0$, for some $r_0 > 0$. We shall be interested in finding solutions of (4.12) near x_0. In order to simplify our notation we shall assume $x_0 = 0$.

If it is easy to change (4.12) into an equivalent equation with a regular singular point at the origin. We let $t = x - x_0$, and

$$\overline{a}(t) = a(x_0 + t) = \alpha_k t^k, \qquad \overline{b}(t) = b(x_0 + t) = \beta_k t^k.$$

The power series for $\overline{a}, \overline{b}$ converge on the interval $|t| < r_0$ about $t = 0$. Let ϕ be any solution of (4.12), and define $\overline{\phi}$ by

$$\overline{\phi}(t) = \overline{\phi}(x_0 + t).$$

Then

$$\frac{d\overline{\phi}}{dt}(t) = \frac{d\phi}{dx}(x_0 + t), \qquad \frac{d^2\overline{\phi}}{dt^2}(t) = \frac{d^2\phi}{dx^2}(x_0 + t),$$

and we see that $\overline{\phi}$ satisfies

$$t^2u'' + \overline{a}(t)tu' + \overline{b}(t)u = 0, \tag{4.13}$$

Where now $u' = du/dt$. This is an equation with a regular singular point at $t = 0$. Conversely, if $\overline{\phi}$ satisfies (4.13) the function ϕ given by

$$\phi(x) = \overline{\phi}(x - x_0)$$

satisfies (4.12). In this sense (4.13) is equivalent to (4.12). With $x_0 = 0$ in (4.12) we may write (4.12) as

$$L(y) = x^2y'' = a(x)xy' + b(x)y = 0 \tag{4.14}$$

Where a, b are analytic at the origin, and have power series expansions

$$a(x) = \sum_{k=0}^{\infty} \alpha_k x^k, \qquad b(x) = \sum_{k=0}^{\infty} \beta_k x^k, \tag{4.15}$$

which are convergent on an interval $|x| < r_0,\ r_0 > 0$. The Euler equation is the special ease of (4.14) with a, b constant. The effect of the higher order terms (terms with x as a factor) in the series (4.15) is to introduce series into the solutions of (4.14). We illustrate by an example.

Example 4.5. Consider the equation

$$L(y) = x^2y'' + \frac{3}{2}xy' + xy = 0 \tag{4.16}$$

which has a regular singular point at the origin. Let us restrict our attention to $x > 0$. Since it is not an Euler equation we can not expect it to have a solution of the form x^r there. However we try for a solution.

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k = c_0x^r + c_1x^{r+1} + \cdots, \quad (c_0 \neq 0), \tag{4.17}$$

We find that $\phi'(x) = c_0rx^{r-1} + c_1(r+1)x^r + c_2(r+2)x^{r+1} + \cdots$,

$$\phi''(x) = c_0r(r-1)x^{r-2} + c_1r(r+1)x^{r-1} + c_2(r+2)(r+1)x^r + \cdots,$$

and hence

$$x^2\phi''(x) = c_0r(r-1)x^r + c_1r(r+1)x^{r+1} + c_2(r+2)(r+1)x^{r+2} + \cdots,$$

$$\tfrac{3}{2}x\phi'(x) = \tfrac{3}{2}c_0rx^r + \tfrac{3}{2}c_1(r+1)x^{r+1} + \tfrac{3}{2}c_2(r+2)x^{r+2} + \cdots,$$

$x\phi(x) = c_0x^{r+1} + c_1x^{r+2} + c_2x^{r+3} + \cdots,$
Then

$$\begin{aligned} L(\phi)(x) &= x^2\phi''(x) + \frac{3}{2}x\phi'(x) + x\phi(x) \\ &= \left[r(r-1) + \frac{3}{2}r\right] c_0x^r + \left\{\left[r(r+1) + \frac{3}{2}(r+1)\right] c_1 + c_0\right\} x^{r+1} \\ &\quad + \left\{\left[(r+2)(r+1) + \frac{3}{2}(r+2)\right] c_2 + c_1\right\} x^{r+2} + \cdots . \end{aligned}$$

If we let $q(r) = r(r-1) + \frac{3}{2}r = r(r + \frac{1}{2})$,

This may be written as

$$\begin{aligned} L(\phi)(x) &= q(r)c_0x^r + \{q(r+1)c_1 + c_0\}x^{r+1} + \{q(r+2)c_2 + c_1\}x^{r+2} + \cdots . \\ &= q(r)c_0x^r + x^r\sum_{k=1}^{\infty}[q(r+k)c_k + c_{k-1}]x^k. \end{aligned}$$

If ϕ is to satisfy $L(\phi)(x) = 0$ all coefficients of the powers of x must vanish. Since we assumed $c_0 \neq 0$ this implies

$$q(r) = 0, \qquad q(r+k)c_k + c_{k-1} = 0, \qquad (k = 1, 2, \cdots). \tag{4.18}$$

The polynominl q is called the *indicial polynomial* for (4.16). It is the coefficient of the lowest power of x appearing in $L(\phi)(x)$, and from (4.18) we see that its roots are the only permissible values of r for which there are solutions of the form (4.17). Here the roots are $r_1 = 0,\ r_2 = \frac{-1}{2}$.

The second set of equations in (4.18) delimits $c_1, c_2, \cdots$ in terms of c_0 and r. If $q(r = k) \neq 0$ for $k = 1, 2, \cdots$, then

$$c_k = \frac{-c_{k-1}}{q(r+k)}, \qquad (k = 1, 2, \cdots).$$

Then

$$\begin{aligned} k = 1, \quad c_1 &= \frac{-c_0}{q(r+1)} \\ k = 2, \quad c_2 &= \frac{-c_1}{q(r+2)} = \frac{c_0}{q(r+1)q(r+2)} \\ k = 3, \quad c_3 &= \frac{-c_2}{q(r+3)} = \frac{-c_0}{q(r+1)q(r+2)q(r+3)} \end{aligned}$$

$$k = n, \quad c_n = \frac{(-)^n c_{n-1}}{q(r+n)} = \frac{(-)^n c_0}{q(r+1)q(r+2)\cdots q(r+n)}$$

In general, for k we have $c_k = \dfrac{(-)^k c_0}{q(r+1)q(r+2)\cdots q(r+k)} \qquad (k = 1, 2, \cdots)$

If $r_1 = 0$, then $q(r_1 + k) = q(k) \neq 0$ for $k = 1, 2, \cdots$.

If $r_2 = \frac{-1}{2}$, then $q(r_2 + k) = q(\frac{-1}{2} + k) \neq 0$ for $k = 1, 2, \cdots$.

Now $\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k = c_0 x^r + \sum_{k=1}^{\infty} c_k x^{r+k} = c_0 x^r + \sum_{k=1}^{\infty} \dfrac{(-)^k c_0}{q(r+1)q(r+2)\cdots q(r+k)} x^{r+k}$

Letting $c_0 = 1$ and $r = r_1 = 0$ we obtain, a solution ϕ_1 given by

$$\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-)^k x^k}{q(1)q(2)\cdots q(k)}$$

Letting $c_0 = 1$ and $r = r_2 = \frac{-1}{2}$ we obtain, a solution ϕ_2 given by

$$\phi_2(x) = x^{-1/2} + x^{-1/2} \sum_{k=1}^{\infty} \frac{(-)^k x^k}{q(k-\frac{1}{2})q(k-\frac{3}{2})\cdots q(\frac{1}{2})}$$

These functions ϕ_1, ϕ_2 will be solutions provided the series converge on some interval containing $x = 0$. Let us write the series for ϕ_1 in the form

$$\phi_1(x) = \sum_{k=0}^{\infty} d_k(x).$$

Using the ratio test we obtain

$$\left| \frac{d_{k+1}(x)}{d_k(x)} \right| = \frac{|x|}{|q(k+1)|} = \frac{|x|}{(k+1)(k+\frac{1}{2})} \to 0 \text{ as } k \to \infty$$

provided $|x| \to \infty$. Thus the series defining ϕ_1 is convergent for all finite x. The same can be shown to hold for the series multiplying $x^{-1/2}$ in the expression for ϕ_2. Thus ϕ_1, ϕ_2 are solutions of (4.16) for all $x > 0$.

To obtain solutions for $x < 0$ we note that all the above computations go through if x^r is replaced everywhere by $|x|^r$, where

$$|x|^r = e^{r \log |x|}. \tag{4.19}$$

Thus two solutions of (4.16) which are valid for all $x \neq 0$ are given by

$$\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-)^k x^k}{q(1)q(2)\cdots q(k)}$$

and $$\phi_2(x) = x^{-1/2}\left[1 + \sum_{k=1}^{\infty} \frac{(-)^k x^k}{q(k-\frac{1}{2})q(k-\frac{3}{2})\cdots q(\frac{1}{2})}\right].$$

Note that the definition (4.19) implies that $|x|^{1/2}$ is the positive square root of $|x|$. It is clear that ϕ_1, ϕ_2 are linearly independent on any interval not containing $x = 0$. For, let $x_0 = 0$. Then $\phi_1(x_0) = 1, \quad \phi_2(x_0) = 0$ and $\phi_1'(x_0) = 0, \quad \phi_2'(x_0) = 1$.

Therefore $W(\phi_1, \phi_2) = 1 \neq 0$ and so ϕ, ϕ_2 are linearly independent.

Exercise:

1. Find the singular points of the following equations, and determine those which are regular singular points:

(a) $x^2y'' + (x + x^2)y' - y = 0$ (b) $3x^2y'' - 5y' + 3x^2y = 0$

(c) $(1 - x^2)y'' - 2xy' + 2y = 0$ (d) $xy'' + 4y = 0$

2. Compute the indicial polynomials and their roots for the following equations:

(a) $x^2y'' + (x + x^2)y' - y = 0$ (b) $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$

4.4 Second order equations with regular singular points - the general case

Theorem 4.6. *Consider the equation*

$$x^2y'' + a(x)xy' + b(x)y = 0,$$

where a, b have convergent power series expansions for $|x| < r_0, \ r_0 > 0$. Let $r - 1, r_2$ $(Rer_1 \geq Rer_2)$ be the root of the roots of the indicial polynomial

$$q(r) = r(r-1) + a(0)r + b(0).$$

For $0 < |x| < r_0$ there is a solution ϕ_1 of the form

$$\phi_1(x) = |x|^{r_1} \sum_{k=0}^{\infty} c_k x^k \qquad (c_0 = 1),$$

where the series converges for $|x| < r_0$. If $r_1 - r_2$ is not zero, or a positive integer then there is a second solution ϕ_2 for $0 < |x| < r_0$ of the form

$$\phi_2(x) = |x|^{r_2} \sum_{k=0}^{\infty} \overline{c}_k x^k \qquad (c_0 = 1),$$

where the series converges for $|x| < r_0$.

The coefficients $c_k, \overline{c}_k$, can be obtained by substitution of the solutions into the *diferential equation.*

Proof. Consider the equation

$$x^2y'' + a(x)xy' + b(x)y = 0, \tag{4.20}$$

Suppose we have a solution ϕ if the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k \quad (c_0 \neq 0,) \tag{4.21}$$

for the equation (4.20) where

$$a(x) = \sum_{k=0}^{\infty} \alpha_k x^k, \qquad b(x) = \sum_{k=0}^{\infty} \beta_k x^k \tag{4.22}$$

for $|x| < r_0$. Then

$$\begin{aligned}
\phi'(x) &= \sum_{k=0}^{\infty}(k+r)c_k x^{k+r-1} = x^{r-1}\sum_{k=0}^{\infty}(k+r)c_k x^k \\
\phi''(x) &= \sum_{k=0}^{\infty}(k+r)(k+r-1)c_k x^{k+r-2} \\
&= x^{r-2}\sum_{k=0}^{\infty}(k+r)(k+r-1)c_k x^k \\
b(x)\phi(x) &= \left(\sum_{k=0}^{\infty}\beta_k x^k\right) x^r \left(\sum_{k=0}^{\infty}\alpha_k x^k\right) = x^r \sum_{k=0}^{\infty}\overline{\beta}_k x^k, \\
&\qquad \text{where } \overline{\beta}_k = \sum_{j=0}^{k} c_j \beta_{k-j} \\
xa(x)\phi'(x) &= x\left(\sum_{k=0}^{\infty}\alpha_k x^k\right) x^r \left(\sum_{k=0}^{\infty}(k+r)c_k x^k\right) = x^r \sum_{k=0}^{\infty}\overline{\alpha}_k x^k, \\
&\qquad \text{where } \overline{\alpha}_k = \sum_{j=0}^{k}(j+r)c_j\alpha_{k-j} \\
x^2\phi''(x) &= x^r \sum_{k=0}^{\infty}(k+r)(k+r-1)c_k x^k.
\end{aligned}$$

Then $L(\phi)(x) = x^r \sum_{k=0}^{\infty} \left[(k+r)(k+r-1)c_k + \overline{\alpha}_k + \overline{\beta}_k\right] x^r$,

and we must have $\left[(k+r)(k+r-1)c_k + \overline{\alpha}_k + \overline{\beta}_k\right] = 0, \quad k = 0, 1, 2, \cdots$.

Then using the definition of $\overline{\alpha}_k,\ \overline{\beta}_k$, we can write

$(k+r)(k+r-1)c_k + \sum_{j=0}^{k}(j+r)c_j\alpha_{k-j} + \sum_{j=0}^{k} c_j\beta_{k-j} = 0$

$[(k+r)(k+r-1) + (k+r)\alpha_0 + \beta_0]\, c_k + \sum_{j=0}^{k-1} [(j+r)\alpha_{k-j} + \beta_{k-j}]\, c_j = 0.$

Then for $k = 0$ we must have

$$r(r-1) + r\alpha_0 + \beta_0 = 0, \tag{4.23}$$

since $c_0 \neq 0$. The second degree polynomial q given be

$$q(r) = r(r-1) + r\alpha_0 + \beta_0$$

is called the *indicial polynomial* for (4.20), and the only admissible values of r are the roots of q. We see that

$$q(k+r)c_k + d_k = 0 \tag{4.24}$$

where

$$d_k = \sum_{j=0}^{k-1}(j+r)c_j\alpha_{k-j} + \sum_{j=0}^{k-1} c_j\beta_{k-j}, \quad k = 1, 2, \cdots . \tag{4.25}$$

Note that d_k is a linear combination of $c_0, c_1, \cdots, c_{k-1}$ with coefficients involving the known functions a, b, and r. Leaving r and c_0 indeterminant for the moment we solve the equations (4.24), (4.25) successively in terms of c_0 and r. The solutions we denote by $C_k(r)$, and the corresponding d_k by $D_k(r)$. Thus

$$D_1(r) = (r\alpha_1 + \beta_1)c_0, \qquad C_1(r) = \frac{-D_1(r)}{q(r+1)},$$

and in general

$$D_k(r) = \sum_{j=0}^{k-1} [(j+r)\alpha_{k-j} + \beta_{k-j}]\, C_j(r), \tag{4.26}$$

$$C_k(r) = \frac{-D_k(r)}{q(r+k)}, \qquad (k = 1, 2, \cdots). \tag{4.27}$$

The C_k thus determined are rational functions of r (quotients of polynomials), and the only points where they cease to exist are the points r for which $q(r+k)=0$ for some $k=1,2,\cdots$. Only two such possible points exist. Let us define Φ by

$$\Phi(x,r)=c_0x^r+x^r\sum_{k=1}^{\infty}C_k(r)x^k. \tag{4.28}$$

If the series in (4.28) converges for $0<x<r_0$, then clearly

$$L(\Phi)(x,r)=c_0q(r)x^r. \tag{4.29}$$

We have now arrived at the following situation. If the ϕ given by (4.21) is a solution of (4.20) then r must be a root of the indicial polynomial q, and the c_k $(k\geq 1)$ are determined uniquely in terms of r and c_0 to be the $C_k(r)$ of (4.27), provided $q(r+k)\neq 0,\quad k=1,2,\cdots$. Conversely, if r is a root of q and if the $C_k(r)$ can be determined (that is, $q(r+k)\neq 0$ for $k=1,2,\cdots$) then the function ϕ given by $\phi(x)=\Phi(x,r)$ is a solution of (4.20) for any choice of c_0, provided the series in (4.28) can be shown to be convergent. 1(z) = z1 Let r_1,r_2 be the two roots of q, and suppose we have labeled them so that $\mathrm{Re}r_1\geq\mathrm{Re}r_2$. Then $q(r_1+k)\neq 0$ for any $k=1,2,\cdots$. Thus $C_k(r_1)$ exists for all $k=1,2,\cdots$, and letting $c_0=C_0(r_1)=1$ we see that the function ϕ_1 given by

$$\phi_1(x)=x^{r_1}\sum_{k=0}^{\infty}C_k(r_1)x^k,\qquad (C_0(r_1)=1),$$

is a solution of (4.20), provided the series is convergent.

If r_1 is a root of q distinct from r_1, and $q(r_2+k)\neq 0$ for $k=1,2,\cdots$, then clearly $C_k(r_2)$ is defined for $k=1,2,\cdots$, and the function ϕ_2 given by

$$\phi_2(x)=x^{r_2}\sum_{k=0}^{\infty}C_k(r_2)x^k,\qquad (C_0(r_2)=1),$$

is another solution of (4.20), provided the series is convergent. The condition

$$q(r+k)\neq 0 \text{ for } k=1,2,\cdots$$

is the same as $r_1\neq r_2+k$ for $k=1,2,\cdots$, or r_1-r_2 is not a positive integer.

Noting that since $\alpha_0=a(0),\ \beta_0=b(0)$, the indicial polynomial q can be written as $q(r)=r(r-1)+a(0)r+b(0)$. $\square$

4.5 The exceptional cases

We divide the exceptional cases into two groups according as the roots $r_1, r_2(\mathrm{Re} r_1 \geq \mathrm{Re} r_2)$ of the indicial polynomial satisfy

$(i) r_1 = r_2$ $\qquad$ $(ii) r_1 - r_2$ is a positive integer.

We try to find solutions for $0 < x < r_0$. For such x we have from (4.28), (4.29)

$$\Phi(x,r) = c_0 x^r + x^r \sum_{k=1}^{\infty} C_k(r) x^k. \tag{4.30}$$

where Φ is given by

$$L(\Phi)(x,r) = c_0 q(r) x^r. \tag{4.31}$$

The $C_k(r)$ are determined recursively by the formulas

$$\begin{aligned} C_0(r) &= c_0 \neq 0 \\ q(r+k)C_k(r) &= -D_k(r), \end{aligned} \tag{4.32}$$

where $D_k(r) = \sum_{j=0}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j}\right] C_j(r), \quad (k = 1, 2, \cdots);$

In case (i) we have $q(r_1) = 0,\ q'(r_1) = 0$, and this suggests formally differentiating (4.29) with respect to r. We obtain

$$\frac{\partial}{\partial r} L(\Phi)(x,r) = L\left(\frac{\partial \Phi}{\partial r}\right)(x,r) = c_0 \left[q'(r) + (\log x) q(r)\right] x^r$$

and we see that if $r = r_1 = r_2,\ c_0 = 1$, then

$$\phi_2(x) = \frac{\partial \Phi}{\partial r}(x, r_1)$$

will yield a solution of our equation, provided the series involved converge. Computing formally from (4.28) we find

$$\begin{aligned} \phi_2(x) &= x^{r_1} \sum_{k=0}^{\infty} C'_k(r_1) x^k + (\log x) x^{r_1} \sum_{k=0}^{\infty} C_k(r_1) x^k \\ &= x^{r_1} \sum_{k=0}^{\infty} C'_k(r_1) x^k + (\log x) \phi_1(x) \end{aligned}$$

where ϕ_1 is the solution already obtained:

$$\phi_1(x) = x^{r_1} \sum_{k=0}^{\infty} C_k(r_1)x^k, \qquad (C_0(r_1) = 1.$$

Note that $C_k'(r_1)$ exists for all $k = 0, 1, 2, \cdots$, since C_k is a rational function of r whose denominator is not zero at $r = r_1$. Also $C_0(r) = 1$ implies that $C_0'(r_1) = 0$, and thus the series multiplying x^{r_1} in ϕ_2 starts with the first power of x.

Let us now turn to the case (ii), and suppose that $r_1 = r_2 + m$, where m is a positive integer. If c_0 is given,

$$C_1(r_2), \cdots, c_{m-1}(r_2)$$

all exists as finite numbers, but since

$$q(r+m)C_m(r) = -D_m(r), \tag{4.33}$$

we run into trouble in trying to compute $C_m(r_1)$. Now $q(r) = (r - r_1)(r - r_2)$, and hence $q(r+m) = (r - r_2)(r + m - r_2)$.

If $D_m(r)$ also has $r - r_2$, as a factor (i.e., $D_m(r_1) = 0$) this would cancel the same factor in $q(r+m)$, and (4.33) would give $C_m(r_1)$ as a finite number. Then $c_{m+1}(r_2), c_{m+2}(r_2), \cdots$ all exists. In this rather special situation we will have a solution ϕ_2 of the form

$$\phi_2(x) = x^{r_2} \sum_{k=0}^{\infty} C_k(r_2)x^k, \qquad (C_0(r_2) = 1).$$

We can always arrange it so that $D_m(r_1) = 0$ by choosing $C_0(r) = r - r_2$.

From (4.32) we see that $D_k(r)$ is linear homogeneous in $C_0(r), \cdots, C_{k-1}(r)$ and hence $D_k(r)$ has $C_0(r) = r - r_2$ as a factor. Thus $C_m(r_2)$ will exists as a finite number. Letting

$$\psi(x,r) = x^r \sum_{k=0}^{\infty} C_k(r)x^k, \qquad (C_0(r) = r - r_2), \tag{4.34}$$

we find formally that

$$L(\psi)(x,r) = (r - r_2)q(r)x^r \tag{4.35}$$

Putting $r = r_2$ we obtain formally a solution ψ given by $\psi(x) = \Psi(x, r_2)$.

However $C_0(r_2) = C_1(r_2) = \cdots = C_{m-1}(r_2) = 0$. Thus the series for ψ actually starts with the $m-$th power of x, and hence ψ has the form

$$\psi(x) = x^{r_2+m}\sigma(x) = x^{r_1}\sigma(x),$$

where σ is some power series. It is not difficult to see that ψ is just a constant multiple of the solution ϕ_1 already obtained. To get a solution really associated with r_2, we differentiate (4.35) with respect to r, obtaining

$$\begin{aligned}\frac{\partial}{\partial r}L(\Psi)(x,r) &= L\left(\frac{\partial\Psi}{\partial r}\right)(x,r)\\ &= q(r)x^r+(r-r_2)\left[q'(r)+(\log x)q(r)\right]x^r.\end{aligned}$$

Now letting $r=r_2$ we find that the ϕ_2 given by

$$\phi_2(x)=\frac{\partial\Psi}{\partial r}(x,r_2)$$

is a solution, provided the series involved are convergent. It has the form

$$\phi_2(x)=x_2^r\sum_{k=0}^{\infty}C_k'(r_2)x^k+(\log x)x_2^r\sum_{k=0}^{\infty}C_k(r_2)x^k,$$

where $C_k(r)=r-r_2$. Since $C_0(r_2)=\cdots=C_{m-1}(r_2)=0$, we may write this as

$$\phi_2(x)=x_2^r\sum_{k=0}^{\infty}C_k'(r_2)x^k+c(\log x)\phi_1(x),$$

where c is some constant.

The method used in this section to obtain solutions is called the *Frobenius method.* All the series obtained converge for $|x|<r_0$. Similarly the solutions for $x<0$ can be obtained by replacing $x^{r_1},\ x^{r_2},\log x$ everywhere by $|x|^{r_1},|x|^{r_2},\log|x|$ respectively.

Theorem 4.7. *Consider the equation*

$$x^2y''+a(x)xy'+b(x)y=0,$$

where a,b have convergent power series expansions for $|x|<r_0,\ r_0>0$. Let $r-1,r_2$ ($Rer_1\geq Rer_2$) be the root of the roots of the indicial polynomial

$$q(r)=r(r-1)+a(0)r+b(0).$$

If $r_1=r_2$ there are two linearly independent solutions ϕ_1,ϕ_2 for $0<|x|<r_0$ of the form

$$\phi_1(x)=|x|^{r_1}\sigma_1(x),\qquad \phi_2(x)=|x|^{r_1+1}\sigma_2(x)+(\log|x|)\phi_1(x),$$

where σ_1,σ_2 have power series expansions which are convergent for $|x|<r_0$ and $\sigma_1(0)\neq 0$.

If r_1-r_2 is a positive integer then there are two linearly independent solutions ϕ_1,ϕ_2 for $0<|x|<r_0$ of the form

$$\phi_1(x)=|x|^{r_1}\sigma_1(x),\qquad \phi_2(x)=|x|^{r_2}\sigma_2(x)+c(\log|x|)\phi_1(x),$$

where σ_1,σ_2 have power series expansions which are convergent for $|x|<r_0$ and

$\sigma_1(0) \neq 0, \sigma_2(0) \neq 0$, *and* c *is a constant. It may happens that* $c = 0$.

Exercise:

1. Obtain two linearly independent solutions of the following equations which are valid near $x = 0$:

(a) $x^2y'' + 3xy' + (1 + x)y = 0$ (b) $x^2y'' + 2x^2y' - 2y = 0$

(c) $3x^2y'' + 5xy' + 3xy = 0$ (d) $x^2y'' + xy' + x^2y = 0$

4.6 The Bessel equation

If α is a constant, Re $\alpha \geq 0$, the *Bessels equation of order* α is the equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0.$$

This has the form

$$x^2y'' + a(x)xy' + b(x)y = 0.$$

with $a(x) = 1,\ \ b(x) = x^2 - \alpha^2$. Since a, b s are analytic at $x = 0$, the Bessel equation has the origin as a regular singular point. The indicial polynomial q is given by

$$q(r) = r(r - 1) + a(0)r + b(0) = r(r - 1) + r - \alpha^2 = r^2 - \alpha^2.$$

whose two roots r_1, r_2 are

$$r_1 = \alpha, \quad r_2 = -\alpha.$$

We shall construct solutions for $x > 0$.

Bessel equation of order zero

Let us consider the case $\alpha = 0$ first. Since the roots are both equal to zero in this case it follows from Theorem 4.7 that there are two solutions ϕ_1, ϕ_2 of the form

$$\phi_1(x) = \sigma_1(x), \qquad \phi_2(x) = x\sigma_2(x) + (\log x)\phi_1(x),$$

where σ_1, σ_2 have power series expansions which converge for all finite x. Let us compute σ_1, σ_2.

Now consider $L(y) = x^2y'' + xy' + x^2y$ and suppose

$$\sigma_1(x) = \sum_{k=0}^{\infty} c_k\ x^k, \qquad (c_0 \neq 0)$$

.

We find

$$
\begin{aligned}
\sigma_1'(x) &= \sum_{k=1}^{\infty} k\ c_k\ x^{k-1} \\
\sigma_1''(x) &= \sum_{k=2}^{\infty} k\ (k-1)\ c_k\ x^{k-2}
\end{aligned}
$$

and hence

$$
\begin{aligned}
x^2\ \sigma_1''(x) &= \sum_{k=2}^{\infty} k\ (k-1)\ c_k\ x^k \\
x\ \sigma_1'(x) &= \sum_{k=1}^{\infty} k\ c_k\ x^k = c_1\ x + \sum_{k=2}^{\infty} k\ c_k\ x^k \\
x^2\ \sigma_1(x) &= \sum_{k=0}^{\infty} c_k\ x^{k+2} = \sum_{k=2}^{\infty} c_{k-2}\ x^k
\end{aligned}
$$

Thus $L(\sigma_1(x)) = \sum_{k=2}^{\infty} [(k\ (k-1)+k)\, c_k + c_{k-2}]\, x^k + c_1\ x = 0.$

Then we see that $c_1 = 0$, and $(k\ (k-1)+k)\, c_k + c_{k-2} = 0$ for $k = 2, 3, \cdots.$

Then $c_k = \dfrac{-c_{k-2}}{k\ (k-1)+k} = \dfrac{-c_{k-2}}{k^2}$ for $k = 2, 3, \cdots.$

Let $c_0 = 1$. Then for $k = 2, 3, \cdots$ we have

$$
\begin{aligned}
k = 2, \qquad c_2 &= \frac{-c_0}{2^2} = \frac{-1}{2^2} \\
k = 4, \qquad c_4 &= \frac{-c_2}{2^2} = \frac{1}{2^2\ 4^2} \\
k = 6, \qquad c_6 &= \frac{-c_4}{2^2} = \frac{-1}{2^2\ 4^2\ 6^2}
\end{aligned}
$$

In general, $c_{2m} = \dfrac{(-1)^m}{2^2\ 4^2\ 6^2\ \cdots (2m)^2}$ for $m = 1, 2, \cdots.$

Since $c_1 = 0$, we have $c_3 = c_5 = \cdots = 0$. Thus $\sigma_1(x)$ contains even powers of x and we obtain

$$
\sigma_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m\ x^{2m}}{2^{2m}\ (m!)^2},
$$

where as usal $0! = 1$, and $2^0 = 1$. The function defined by this series is called the

Bessel functionm of eero order of the first kind and is denoted by J_0. Thus

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \; x^{2m}}{2^{2m} \; (m!)^2} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$$

Note: (Ratio test) Suppose we have the series $\sum a_n$. Define $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then if $L < 1$ the series is absolutely convergent (and hence convergent).

It can be checked by the ratio test that this series converges for all finite x. We now determine a second solution ϕ_2 for the Bessel equation of order zero. Letting $\phi_1 = J_0$ this solution has the form

$$\phi_2(x) = \sum_{k=0}^{\infty} c_k \; x^k + (\log x) \; \phi_1(x), \qquad (c_0 = 0).$$

We obtain

$$\phi_2'(x) = \sum_{k=1}^{\infty} k \; c_k \; x^{k-1} + \frac{\phi_1(x)}{x} + (\log x) \; \phi_1'(x),$$

$$\phi_2''(x) = \sum_{k=2}^{\infty} k \; (k-1) \; c_k \; x^{k-2} - \frac{\phi_1(x)}{x^2} + \frac{2}{x}\phi_1'(x) + (\log x) \; \phi_1''(x).$$

Thus

$$\begin{aligned}
L(\phi_2)(x) &= x^2\phi_2''(x) + x\phi_2'(x) + x^2\phi_2(x) \\
&= \sum_{k=2}^{\infty} k \; (k-1) \; c_k \; x^k - \phi_1(x) + 2x\phi_1'(x) + (\log x)x^2\phi_1''(x) \\
&\quad + \sum_{k=1}^{\infty} k \; c_k \; x^k + \phi_1(x) + (\log x)x\phi_1'(x) + \sum_{k=0}^{\infty} c_k \; x^{k+2} + (\log x)x^2\phi_1(x) \\
&= \sum_{k=2}^{\infty} k \; (k-1) \; c_k \; x^k + c_1 x + \sum_{k=2}^{\infty} k \; c_k \; x^k + \sum_{k=2}^{\infty} c_{k-2} \; x^k \\
&\quad + 2x\phi_1'(x) + (\log x)(x^2\phi_1''(x) + x\phi_1'(x) + x^2\phi_1(x)) \\
&= \sum_{k=2}^{\infty} k \; (k-1) \; c_k \; x^k + c_1 x + \sum_{k=2}^{\infty} k \; c_k \; x^k + \sum_{k=2}^{\infty} c_{k-2} \; x^k \\
&\quad + 2x\phi_1'(x) + (\log x)L(\phi_1)(x) \\
&= \sum_{k=2}^{\infty} [(k \; (k-1) + k) \; c_k + c_{k-2}] \; x^k + c_1 x + 2x\phi_1'(x), \quad \text{since } L(\phi_1)(x) = 0
\end{aligned}$$

Since $L(\phi_2)(x) = 0$, we have

$$c_1 x + \sum_{k=2}^{\infty} [k^2 \; c_k + c_{k-2}] \; x^k = -2x\phi_1'(x) = -2 \sum_{m=1}^{\infty} \frac{(-1)^m (2m) x^{2m}}{2^{2m}(m!)^2}$$

Hence by equating coefficient of x we have $c_1 = 0$. Since the series on the right side has only the even powers of x, we have $c_3 = c_5 = \cdots = 0$.

The recursion relation for the other coefficients is

$$(2m)^2\, c_{2m} + c_{2m-2} = \frac{(-1)^{m+1}(m)x^{2m}}{2^{2m-2}(m!)^2}, \quad (m = 2, 3, \cdots)$$

Then we have $c_2 = \dfrac{1}{2^2}$

$$c_4 = \frac{1}{4^2}\left(-\frac{1}{2^2} - \frac{1}{2\;2^2}\right) = -\frac{1}{2^2\;4^2}\left(1 + \frac{1}{2}\right)$$

$$c_6 = \frac{1}{6^2}\left[\frac{1}{2^2\;4^2}\left(1 + \frac{1}{2}\right) + \frac{1}{2^2\;4^2}\frac{1}{3}\right] = \frac{1}{2^2\;4^2\;6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right), \cdots ,$$

and it can be shown by induction that

$$c_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) \text{ where } m = 1, 2, \cdots .$$

The solution thus determined is called a *Bessel function of zero order of the second kind*, and is denoted by K_n. Hence

$$K_n(x) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)\left(\frac{x}{2}^{2m}\right) + (\log x)J_0(x).$$

Using the ratio test it is easy to check that the series on the right is convergent for all finite x.

Bessel equation of order α

Now we compute solutions for the Bessel equation of order α, where $\alpha \neq 0$ and Re $\alpha \geq 0$:

$$L(y) = x^2y'' + xy' + (x^2 - \alpha^2)y = 0.$$

Let $x > 0$. The roots of the indicial polynomial are $r_1 = \alpha$, $\;r_2 = -\alpha$.

First we determine a solution corresponding to the root $r_1 = \alpha$. The solution ϕ_1 has the form

$$\phi_1(x) = x^\alpha \sum_{k=0}^{\infty} c_k\, x^k, \qquad (c_0 \neq 0).$$

That is $\phi_1(x) = \sum_{k=0}^{\infty} c_k\, x^{k+\alpha}$.

$$
\begin{aligned}
\phi_1'(x) &= \sum_{k=0}^{\infty}(\alpha+k)\ c_k\ x^{\alpha+k-1} \\
\phi_1''(x) &= \sum_{k=0}^{\infty}(\alpha+k)\ (\alpha+k-1)\ c_k\ x^{\alpha+k-2} \\
x^2\ \phi_1''(x) &= \sum_{k=0}^{\infty}(\alpha+k)\ (\alpha+k-1)\ c_k\ x^{\alpha+k} \\
x\ \phi_1'(x) &= \sum_{k=0}^{\infty}(\alpha+k)\ c_k\ x^{\alpha+k} \\
(x^2-\alpha^2)\phi_1(x) &= x^2\phi_1(x)-\alpha^2\phi_1(x) \\
&= \sum_{k=0}^{\infty} c_k\ x^{\alpha+k+2}-\alpha^2\sum_{k=0}^{\infty} c_k\ x^{\alpha+k} \\
&= \sum_{k=2}^{\infty} c_{k-2}\ x^{\alpha+k}-\alpha^2\sum_{k=0}^{\infty} c_k\ x^{\alpha+k}
\end{aligned}
$$

Then

$$
\begin{aligned}
L(\phi_1)(x) &= \sum_{k=0}^{\infty}\left[(\alpha+k)(\alpha+k-1)+(\alpha+k)\right]c_k x^{\alpha+k} \\
&\quad +\sum_{k=2}^{\infty} c_{k-2}\ x^{\alpha+k}-\alpha^2\sum_{k=0}^{\infty} c_k\ x^{\alpha+k} \\
&= \sum_{k=0}^{\infty}\left[(\alpha+k)^2-\alpha^2\right]c_k x^{\alpha+k}+\sum_{k=2}^{\infty} c_{k-2}\ x^{\alpha+k} \\
&= (\alpha^2-\alpha^2)c_0x^{\alpha}+((\alpha+1)^2-\alpha^2)c_1x^{\alpha+1} \\
&\quad +\sum_{k=2}^{\infty}\left\{\left[(\alpha+k)^2-\alpha^2\right]c_k+c_{k-2}\right\}x^{\alpha+k}
\end{aligned}
$$

Since $L(\phi_1)(x)=0$, $c_1=0$ and $[(\alpha+k)^2-\alpha^2]\,c_k+c_{k-2}=0$ for $k=2,3,\cdots$.

Therefore $c_k=\dfrac{-c_{k-2}}{(\alpha+k)^2-\alpha^2}=\dfrac{-c_{k-2}}{k(2\alpha+k)}$, since $k(2\alpha+k)\neq 0$ for $k=2,3,\cdots$.

Since $c_1=0$, $c_3=c_5=\cdots=0$. Then for $k=2,3,\cdots$ we have

$$
\begin{aligned}
k = 2, \qquad c_2 &= \frac{-c_0}{2(2\alpha+2)} = \frac{-c_0}{2^2(\alpha+1)} \\
k = 4, \qquad c_4 &= \frac{-c_2}{4(2\alpha+4)} = \frac{c_0}{2^2(\alpha+1)2^3(\alpha+2)} = \frac{c_0}{2^4\ 2!(\alpha+1)(\alpha+2)} \\
k = 6, \qquad c_6 &= \frac{-c_4}{6(2\alpha+6)} = \frac{-c_0}{2^4\ 2!(\alpha+1)(\alpha+2)12(\alpha+3)} = \frac{c_0}{2^6\ 3!(\alpha+1)(\alpha+2)(\alpha+3)}
\end{aligned}
$$

In general, we have $c_{2m} = \dfrac{(-1)^m\ c_0}{2^{2m}\ m!\ (\alpha+1)\cdots(\alpha+m)}$ where $m = 1, 2, \cdots$.

Thus our solutions becomes

$$
\phi_1(x) = c_0 x^\alpha + c_0 x^\alpha \sum_{m=1}^{\infty} \frac{(-1)^m\ x^{2m}}{2^{2m}\ m!\ (\alpha+1)\cdots(\alpha+m)} \tag{4.36}
$$

For $\alpha = 0,\ c_0 = 1$, this reduces to $J_0(x)$.

It is usual to choose

$$
c_0 = \frac{1}{2^\alpha\ \Gamma(\alpha+1)} \tag{4.37}
$$

where Γ is a *gamma function* defined by

$$
\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx, \qquad (\text{Re } z > 0)
$$

Then it is clear that $\Gamma(z+1) = z\Gamma(z)$. Indeed by applying integration by parts, we have $Gamma(z+1) = \int_0^\infty e^{-x} x^z dx$. Taking $u = x^z,\ dv = e^{-x}dx$ we have $du = zx^{z-1}dz,\ \ v = -e^{-x}$.

Then

$$
\begin{aligned}
\Gamma(z+1) &= \left[-x^z e^{-x}\right]_0^\infty + \int_0^\infty e^{-x} z x^{z-1} dx \\
&= 0 + z \int_0^\infty e^{-x} x^{z-1} dx \\
&= z\Gamma(z)
\end{aligned}
$$

Also $\Gamma(1) = 1$. If z is a positive integer n, then $\Gamma(n+1) = n!$.

Thus the gamma function is an extension of the factorial function to numbers which are not integers.

Suppose N is a positive integer such that $-N < Rez < -N+1$. Then $Re(z+N) > 0$ and we can define $\Gamma(z)$ in terms of $\Gamma(z+N)$ by

$$\Gamma(z) = \frac{\Gamma(z+N)}{z(z+1)\cdots(z+N-1)}, \text{ provided } z \neq -N+1$$

.

Now,

$$\begin{aligned}
\phi_1(x) &= c_0 x^\alpha + c_0 x^\alpha \sum_{m=1}^{\infty} \frac{(-1)^m \ x^{2m}}{2^{2m} \ m! \ (\alpha+1)\cdots(\alpha+m)} \\
&= \frac{x^\alpha}{2^\alpha \ \Gamma(\alpha+1)} + \frac{x^\alpha}{2^\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m \ x^{2m}}{2^{2m} \ m! \ (\alpha+1)\cdots(\alpha+m)\Gamma(\alpha+1)}
\end{aligned}$$

Since from the definition of $\Gamma(z)$ we have

$$\Gamma(z) = \frac{\Gamma(z+N)}{z(z+1)\cdots(z+N-1)}$$

Therefore we have

$$\begin{aligned}
\Gamma(\alpha+1) &= \frac{\Gamma(\alpha+1+N)}{(\alpha+1)(\alpha+2)\cdots(\alpha+N)} \\
\Gamma(\alpha+1+N) &= (\alpha+1)(\alpha+2)\cdots(\alpha+N)\Gamma(\alpha+1)
\end{aligned}$$

Hence

$$\begin{aligned}
\phi_1(x) &= \frac{x^\alpha}{2^\alpha \ \Gamma(\alpha+1)} + \frac{x^\alpha}{2^\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m \ x^{2m}}{2^{2m} \ m! \ (\alpha+1)\cdots(\alpha+m)\Gamma(\alpha+1)} \\
&= \frac{x^\alpha}{2^\alpha \ \Gamma(\alpha+1)} + \frac{x^\alpha}{2^\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m \ x^{2m}}{2^{2m} \ m! \ \Gamma(\alpha+m+1)} \\
&= \frac{x^\alpha}{2^\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m \ x^{2m}}{2^{2m} \ m! \ \Gamma(\alpha+m+1)}
\end{aligned}$$

This is denoted by J_α. That is

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \ \Gamma(\alpha+m+1)} \left(\frac{x}{2}\right)^{2m}.$$

Notice that this formula for $J\alpha$. reduces to J_0 when $\alpha = 0$, since $\Gamma(m+1) = m!$.

There are now two cases according as $r_1 - r_2 = 2\alpha$ is a positive integer or not.

Case 1: If $r_1 - r_2 = 2\alpha$ is not a positive integer, by Therem 4.6 there is another solution ϕ_2 of the form

$$\phi_2(x) = x^{-\alpha} \sum_{k=}^{\infty} c_k x^k$$

We find that our calculations for the root $r_2 = -\alpha$ is same as $r_1 = \alpha$ provided that we replace α by $-\alpha$ everywhere. Then

$$J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ \Gamma(m-\alpha+1)} \left(\frac{x}{2}\right)^{2m}$$

gives a second solution in case 2α is not a positive integer.

Case 2: $r_1 - r_2 = 2\alpha$ is a positive integer if α is a positive integer say $\alpha = n$. By Theorem 4.7, there is another solution ϕ_2 of the form

$$\begin{aligned}
\phi_2(x) &= \sum_{k=0}^{\infty} c_k x^{k-n} + c(\log x) J_n(x) \\
\phi_2'(x) &= \sum_{k=0}^{\infty} (k-n) c_k x^{k-n-1} + c(\log x) J_n'(x) + c\frac{J_n(x)}{x} \\
\phi_2''(x) &= \sum_{k=0}^{\infty} (k-n)(k-n-1) c_k x^{k-n-2} + c(\log x) J_n''(x) \\
&\quad + c\frac{J_n'(x)}{x} + c\frac{J_n'(x)}{x} - c\frac{J_n(x)}{x^2} \\
x^2\phi_2''(x) &= \sum_{k=0}^{\infty} (k-n)(k-n-1) c_k x^{k-n} + c(\log x) x^2 J_n''(x) \\
&\quad + cxJ_n'(x) + cxJ_n'(x) - cJ_n(x) \\
x\phi_2'(x) &= \sum_{k=0}^{\infty} (k-n) c_k x^{k-n} + c(\log x) x J_n'(x) + cJ_n(x) \\
(x^2 - n^2)\phi_2(x) &= \sum_{k=0}^{\infty} c_k x^{k-n+2} - n^2 \sum_{k=0}^{\infty} c_k x^{k-n} + c(\log x)(x^2 - n^2) J_n(x)
\end{aligned}$$

$$
\begin{aligned}
L(\phi_2)(x) &= x^2\phi_2''(x) + x\phi_2'(x) + (x^2-n^2)\phi_2(x) \\
&= \sum_{k=0}^{\infty}\left[(k-n)(k-n-1)+(k-n)-n^2\right]c_k x^{k-n} + cx^2 J_n''(x)(\log x) \\
&\quad +2cxJ_n'(x) + cx(\log x)J_n'(x) + \sum_{k=0}^{\infty} c_k x^{k-n+2} + (x^2-n^2)c(\log x)J_n(x) \\
&= (n^2-n^2)c_0x^{-n} + \left[(1-n)^2-n^2\right]c_1x^{1-n} + 2cxJ_n'(x) \\
&\quad +\sum_{k=2}^{\infty}\left[((k-n)^2-n^2)c_k + c_{k-2}\right]x^{k-n} + c(\log x)L(J_n(x)) \\
&= (0)c_0x^{-n} + \left[(1-n)^2-n^2\right]c_1x^{1-n} + 2cxJ_n'(x) \\
&\quad +\sum_{k=2}^{\infty}\left[((k-n)^2-n^2)c_k + c_{k-2}\right]x^{k-n}, \quad \text{since} \quad L(J_n(x)) = 0,
\end{aligned}
$$

Since $L(\phi_2)(x) = 0$, we have on multiplying by x^n

$$
\begin{aligned}
& x^n(0)c_0x^{-n} + \left[(1-n)^2-n^2\right]c_1x^{1-n}x^n + 2cx^{n+1}J_n'(x) \\
&\qquad + x^n\sum_{k=2}^{\infty}\left[((k-n)^2-n^2)c_k + c_{k-2}\right]x^{k-n} = 0
\end{aligned}
$$

$$
(0)c_0 + \left[(1-n)^2-n^2\right]c_1x + \sum_{k=2}^{\infty}\left[((k-n)^2-n^2)c_k + c_{k-2}\right]x^k = -2cx^{n+1}J_n'(x)
$$

Since $J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\;\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n}$, we have

$$
J_n'(x) = \sum_{m=0}^{\infty}\frac{(-1)^m(2m+n)x^{2m+n-1}}{2^{2m+n}m!\;\Gamma(n+m+1)}
$$

Therefore

$$
\begin{aligned}
(0)c_0 + (1-2n)c_1x + \sum_{k=2}^{\infty}\left[k(k-2n)c_k + c_{k-2}\right]x^k &= -2c\sum_{m=0}^{\infty}\frac{(-1)^m(2m+2n)x^{2m+n}}{2^{2m+n}m!\;\Gamma(n+m+1)} \\
&= -2c\sum_{m=0}^{\infty}(2m+n)\;d_{2m}x^{2m+2n}
\end{aligned}
\tag{4.38}
$$

where

$$
d_{2m} = \frac{(-1)^m}{2^{2m+n}m!(m+n)!} \tag{4.39}
$$

since $\Gamma(n+m+1) = (n+m)!$.

The right hand side of above series (4.38) begins with x^{2n}, and since n is a positive integer we have $c_1 = 0$. Further if $n > 1$,

$$k(k-2n)c_k + c_{k-2} = 0, \qquad (k = 2, 3, \cdots, 2n-1),$$

and this implies $c_3 = c_5 = \cdots = c_{2n-1} = 0$, whereas

$$c_2 = \frac{c_0}{2^2(n-1)}, \qquad c_4 = \frac{c_2}{2^4\ 2!(n-1)(n-2)},$$

and in general

$$c_{2j} = \frac{c_0}{2^{2j}\ j!(n-1)(n-2)\cdots(n-j)}, \qquad (j = 1, 2, \cdots, n-1)$$

Comparing the coefficient of x^{2n} in (4.38) we obtain

$$c_{2n-2} = -2cnd_0 = \frac{c}{2^{n-1}(n-1)!}.$$

On the other hand from (4.39) it follows that

$$c_{2n-1} = \frac{c_0}{2^{2n-1}(n-1)!(n-1)!},$$

and therefore

$$c = \frac{c_0}{2^{n-1}(n-1)!}. \tag{4.40}$$

Since the series on the right side of (4.38) contains only even powers of x the same must be true of the series on the left side of (4.38), and this implies

$$c_{2n+1} = c_{2n+3} = \cdots = 0.$$

The coefficient c_{2n} is undetermined, but the remaining coefficients $c_{2n+2}, c_{2n+4}, \cdots$ are obtained from the equations

$$2m(2n+2m)c_{2n+2m} + c_{2n+2m-1} = -2c(n+2m)d_{2m}, \qquad (m = 1, 2, \cdots)$$

For $m = 1$, we have

$$c_{2n+1} = -\frac{cd_2}{2}\left(1 + \frac{1}{n+1}\right) - \frac{c_{2n}}{4(n+1)}$$

We now choose c_{2n} so that

$$\frac{c_{2n}}{4(n+1)} = \frac{cd_2}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

Since $4(n+1)d_2 = -d_0$,

$$c_{2n} = -\frac{cd_0}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

With the choice of c_{2n} we have

$$c_{2n+2} = -\frac{cd_2}{2}\left(1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right)$$

For $m = 2$ we obtain

$$c_{2n+4} = -\frac{cd_0}{2}\left(\frac{1}{2} + \frac{1}{n+2}\right) - \frac{c_{2n+2}}{2^n\ 2\ (n+2)}$$

Since $2^n\ 2\ (n+2)d_4 = -d_2$,

$$\frac{c_{2n+2}}{2^n\ 2\ (n+2)} = \frac{cd_4}{2}\left(1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right)$$

and therefore

$$c_{2n+4} = -\frac{cd_4}{2}\left(1 + \frac{1}{2} + 1 + \frac{1}{2} + \cdots + \frac{1}{n+2}\right)$$

It can be shown by induction that

$$c_{2n+2m} = -\frac{cd_{2m}}{2}\left[\left(1 + \frac{1}{2} + \cdots \frac{1}{m}\right) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+m}\right)\right], \qquad (m = 1, 2, \cdots .)$$

Finally, we obtain for our solution ϕ_2, the function given by

$$\begin{aligned}\phi_2(x) &= c_0x^{-n} + c_0x^{-n}\sum_{j=1}^{\infty}\frac{x^{2j}}{2^{2j}\ j!\ (n-1)\cdots(n-j)} - \frac{cd_0}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)x^n \\ &\quad -\frac{c}{2}\sum_{m=1}^{\infty} d_{2m}\left[\left(1 + \frac{1}{2} + \cdots \frac{1}{m}\right) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+m}\right)\right]x^{n+2m} \\ &\quad +c(\log x)J_n(x),\end{aligned}$$

where c_0 and c are constants related by (4.40), and d_{2m} is given by (4.39). When $c = 1$ the resulting function ϕ_2 is often denoted by K_n. In this case

$$c_0 = -2^{n-1}(n-1)!,$$

and therefore we may write

$$K_n(x) = -\frac{1}{2}\left(\frac{x}{2}\right)^{-n}\sum_{j=0}^{n-1}\frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j} - \frac{1}{2}\frac{1}{n!}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\left(\frac{x}{2}\right)^n$$

$$-\frac{1}{2}\left(\frac{x}{2}\right)^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m!\ (m+n)!}\left[\left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{m+n}\right)\right]\left(\frac{x}{2}\right)^{2m}$$

$$+(\log x)J_n(x)$$

This formula reduces to the one for $K_0(x)$ when $n = 0$, provided we interpret the first two sums on the right as zero in this case. The function K_n is called a *Bessel funclion of order n of the second kind.*

Properties of Bessel function

1. $\frac{d}{dx} J_0(x) = -J_1(x)$.

Proof: We know that

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ \Gamma(\alpha+m+1)} \left(\frac{x}{2}\right)^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ (\alpha+m)!} \left(\frac{x}{2}\right)^{2m+\alpha}$$

Then

$$\begin{aligned} J_0(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ m!} \left(\frac{x}{2}\right)^{2m} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m\ x^{2m}}{m!\ m!\ 2^{2m}} \end{aligned}$$

$$\begin{aligned} J_1(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ (m+1)!} \left(\frac{x}{2}\right)^{2m+1} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m\ x^{2m+1}}{m!\ (m+1)!\ 2^{2m+1}} \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dx}(J_0(x)) &= \sum_{m=1}^{\infty} \frac{(-1)^m\ (2m)\ x^{2m-1}}{m!\ m!\ 2^{2m}} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\ (2(m+1))\ x^{2(m+1)-1}}{(m+1)!\ (m+1)!\ 2^{2(m+1)}} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m\ (-1)\ (2(m+1))\ x^{2m+1}}{m!(m+1)\ (m+1)!\ 2^{2m+2}} \\ &= -\sum_{m=0}^{\infty} \frac{(-1)^m\ x^{2m+1}}{(m)!\ (m+1)!\ 2^{2m+1)}} \\ &= -J_1(x) \end{aligned}$$

2. $J_{-\alpha}(x) = (-1)^\alpha\ J_\alpha(x)$.

Proof: We know that

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ (\alpha+m)!} \left(\frac{x}{2}\right)^{2m+\alpha}$$

$$J_{-\alpha}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ (m-\alpha)!} \left(\frac{x}{2}\right)^{2m-\alpha}$$

For $m = 0, 1, 2, \cdots, \alpha - 1$, $(n-\alpha)!$ is $\pm\infty$. Therefore

$$J_{-\alpha}(x) = \sum_{m=\alpha}^{\infty} \frac{(-1)^m}{m!\ (m-\alpha)!} \left(\frac{x}{2}\right)^{2m-\alpha}$$

Put $m - \alpha = n$,

$$J_{-\alpha}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+\alpha}}{(n+\alpha)!\ n!} \left(\frac{x}{2}\right)^{2n-\alpha}$$

.

That is $$J_{-\alpha}(x) = (-1)^{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\alpha)!\ n!} \left(\frac{x}{2}\right)^{2n-\alpha}.$$

Hence $J_{-\alpha}(x) = (-1)^{\alpha}\ J_{\alpha}(x)$.

Chapter 5

Existence and uniqueness of solutions to first order equations

5.1 Introduction

Consider the general first order equation

$$y' = f(x, y) \tag{5.1}$$

when f is some continuous function. Now we consider one special case namely the *linear equation.*

5.2 Linear equation

Consider the linear equation

$$y' + g(x)y = h(x), \tag{5.2}$$

where g, h are continuous on some interval I. Any solution ϕ if (5.2) can be written in the form

$$\phi(x) = e^{-Q(x)} \int\limits_{x_0}^{x} e^{Q(t)} h(t)dt + ce^{-Q(x)}, \tag{5.3}$$

where $Q(x) = \int\limits_{x_0}^{x} g(t)dt$,

x_0 is in I and c is a constant.

Example 5.1. Consider the linear equation

$$y' + xy = e^{-x^2/2}$$

Here $g(x) = x$ and $h(x) = e^{-x^2/2}$.Then

$$Q(x) = \int g(x)dx = \int xdx = \frac{x^2}{2}$$

Therefore

$$\begin{aligned} y &= e^{-Q(x)} \int e^{Q(x)} h(x)dx + ce^{-Q(x)} \\ &= e^{-x^2/2} \int \left(e^{x^2/2}\right)\left(e^{-x^2/2}\right) dx + ce^{-x^2/2} \\ &= e^{-x^2/2} x + ce^{-x^2/2} \\ y &= e^{-x^2/2}(x + c) \end{aligned}$$

Example 5.2. Consider the linear equation

$$y' - y \cot x = 2x \sin x$$

Here $g(x) = -\cot x$ and $h(x) = 2x \sin x$.Then

$$Q(x) = \int g(x)dx = -\int \cot xdx = -\log(\sin x)$$

Therefore

$$\begin{aligned} y &= e^{-Q(x)} \int e^{Q(x)} h(x)dx + ce^{-Q(x)} \\ &= e^{\log(\sin x)} \int (-\log(\sin x)) (2x \sin x) dx + ce^{\log(\sin x)} \\ &= \sin x \int \left(\frac{1}{\sin x}\right) (2x \sin x) dx + c \sin x \\ &= (\sin x)(x^2) + c \sin x \\ y &= (\sin x)(x^2 + c) \end{aligned}$$

If f is not a linear equation there are certain limitations which must be expected concerning any general existence theorem. To illustrate this consider the equation

$$y' = y^2$$

Here $f(x, y) = y^2$ and we see f has serivatives of all orders with respect to x and

y at every point in the $(x, y)-$ plane. A solution ϕ of this equation satisfying the initial condition $\phi(1) = -1$ is given by $\phi(x) = -\frac{1}{x}$. However this solution ceases to exist at $x = 0$ even though f is a nice function there. This example shows that any general existence theorem for (5.1) can only assert the existence of a solution on some interval near-by the initial point.

The above phenomenon does not occur in the case of the linear equation (5.2), for it is clear from (5.3) that any solution ϕ exists on all of the interval I. This points up one of the fundamental difficulties we encounter when we consider nonlinear equations. The equation often gives no clue as to how far a solution will exist.

We prove that initial value problems for equation (5.1) have unique solutions which can be obtained by an approximation process, provided f satisfies an additional condition, the Lipschitz condition. We first concentrate our attention on the case when f is real-valued, and later show how the results carry over to the situation when f is complex-valued.

Exercise:
Find the solution for the following equation.
(a) $(1 + x^2)y' + y = \tan^{-1} x$ (b) $y' + y \sec x = \tan x$

5.3 Equations with variables separated

A first order equation

$$y' = f(x, y)$$

is said to have the *variables separated* if f can be written in the form

$$f(x, y)\frac{g(x)}{h(y)},$$

where g, h are functions of a single argument. In this case we may write our equation as

$$h(y)\frac{dy}{dx} = g(x) \qquad \text{or} \qquad h(y)dy = g(x)dx \tag{5.4}$$

Let us discuss the equation (5.4) in the case g and h are continuous real-valued functions defined for real x and y respectively. If ϕ is a real-valued solution of (5.4) on some interval I containing a point x_0 then

$$h(\phi(x))\phi'(x) = g(x)$$

for all x in I and therefore

$$\int_{x_0}^{x} h(\phi(t))\phi'(t)dt = \int_{x_0}^{x} g(t)dt \qquad (5.5)$$

for all x in I. Letting $u = \phi(t)$ in the integral on the left in (5.5), we see that (5.5) may be written as

$$\int_{\phi(x_0)}^{\phi(x)} h(u)du = \int_{x_0}^{x} g(t)dt$$

Conversely, suppose x and y are related by the formula

$$\int_{y_0}^{y} h(u)du = \int_{x_0}^{x} g(t)dt \qquad (5.6)$$

and that this defines implicitly a differentiable function ϕ for x in I. Then this function satisfies

$$\int_{y_0}^{\phi(x)} h(u)du = \int_{x_0}^{x} g(t)dt$$

for all x in I, and differentiating we obtain $h(\phi(x))\phi'(x) = g(x)$, which shows that ϕ is a solution of (5.4) on I.

The usual way of dealing with (5.4) is to write it as $h(y)dy = g(x)dx$ (thus seperating the variables) and then integrate to obtain

$$\int h(y)dy = \int g(x)dx + c,$$

where c is a constant and the integrals are anti-derivatives. Thus

$$H(y) = \int h(y)dy, \qquad G(x) = \int g(x)dx,$$

represent any two functions H, G such that $H' = h$ and $G' = g$. Then any differentiable function ϕ which is defined implicitely by the relation

$$H(y) = G(x) + c \qquad (5.7)$$

will be the solution of (5.4). We summarize in the following theorem.

Theorem 5.3. *Let g, h be continuous real-valued functions for $a \leq x \leq b$, $c \leq y \leq d$ respectively and consider the equation*

$$h(y)y' = g(x)$$

If G, H are any functions such that $G' = g$ and $H' = h$, and c is any constant *such that the relation*

$$H(y) = G(x) + c$$

defines a real-valued differentiable function ϕ, for x in some interval I contained in $a \leq x \leq b$, then ϕ will be a solution of (5.4) on I. Conversely, if ϕ is a solution of (5.4) on I, it satisfies the relation $H(y) = G(x) + c$ on I for some constant c.

Remark 5.4. Consider the equation $y' = \dfrac{g(x)}{h(y)}$

Case 1: Let $h(y) = 1$. Then

$$\begin{aligned} y' &= g(x) \\ \frac{dy}{dx} &= g(x) \\ dy &= g(x)dx \\ y &= \int g(x)dx \end{aligned}$$

Every solution ϕ has the form $\phi(x) = G(x) + c$ where $G(x) = \int g(x)dx$ and c is a constant.

Case 2: Let $g(x) = 1$. Then

$$\begin{aligned} y' &= \frac{1}{h(y)} \\ \frac{dy}{dx} &= \frac{1}{h(y)} \\ dx &= h(y)dy \\ x + c &= \int h(y)dy \end{aligned}$$

Every solution ϕ has the form $H(y) = x + c$ where $H(y) = \int h(y)dy$ and c is a constant.

Example 5.5. Consider the equation $y' = y^2$. Here $h(y) = \dfrac{1}{y^2}$, which is not continuous at $y = 0$. We have

$$\begin{aligned} \frac{dy}{y^2} &= dx \\ \int \frac{dy}{y^2} &= \int dx \end{aligned}$$

$$
\begin{aligned}
-\frac{1}{y} &= x + c \\
y &= \frac{-1}{x + c}
\end{aligned}
$$

This if c is any constant, the function ϕ is given by

$$\phi(x) = \frac{-1}{x + c}$$

is a solution of the equation $y' = y^2$ provided $x \neq c$.

Remark 5.6. It is important to remark that the separation of variables method of finding solutions may not yield all solutions of an equation.

For example, it is clear from the above example that the function ψ which is identically zero for all x is a solution of the equation. However, for no constant c will the ϕ yield this solution.

Example 5.7. Consider the equation $y' = 3y^{2/3}$.

Here $h(y) = \dfrac{1}{3y^{2/3}}$, which is not continuous at $y = 0$. We have

$$
\begin{aligned}
\frac{dy}{y^{2/3}} &= dx \\
\int \frac{dy}{y^{2/3}} &= \int dx \\
y^{1/3} &= x + c \\
y &= (x + c)^3
\end{aligned}
$$

This if c is any constant, the function ϕ is given by

$$\phi(x) = (x + c)^3$$

is a solution of the equation $y' = y^{2/3}$ for any constant c

Exercise:

1. Find all real-valued solutions of the following equations:

(a) $y' = x^2 y$ (b) $yy' = x$

(c) $y' = \dfrac{x + x^2}{y - y^2}$ (d) $y' = x^2 y^2 - 4x^2$

Homogeneous equation

Definition 5.8. A function f defined for real x, y is said to be homogeneous of degree k if

$$f(tx, ty) = t^k\ f(x, y) \text{ for all } t, x, y.$$

Note: If f is a homogeneous function of degree zero then we have $f(tx, ty) = f(x, y)$.

Definition 5.9. The equation $y' = f(x, y)$ is homogeneous if f is a homogeneous function of degree zero.

We consider a equation of the form

$$y' = f(x, y) = \frac{g(x, y)}{h(x, y)}$$

where g, h are homogeneous functions of same degree. This equation can be reduced to ones with variables separated.

To see this, let $y = vx$ in $y' = f(x, y)$. Then we obtain

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = f(x, y) \Rightarrow v + x\frac{dv}{dx} = f(x, vx) = f(1, v)$$

Hence $v' = \dfrac{dv}{dx} = \dfrac{f(1, v) - v}{x}$ which is an equation for v with variables separated. Then we obtain final solution by replacing v by $\dfrac{y}{x}$.

Example 5.10. Consider the equation $y' = \dfrac{x + y}{x - y}$

Let $y = vx$. Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\begin{aligned}
v + x\frac{dv}{dx} &= \frac{x + vx}{x - vx} \\
v + x\frac{dv}{dx} &= \frac{1 + v}{1 - v} \\
x\frac{dv}{dx} &= \frac{1 + v}{1 - v} - v \\
x\frac{dv}{dx} &= \frac{1 + v^2}{1 - v} \\
\frac{1 - v}{1 + v^2}dv &= \frac{dx}{x} \\
\frac{1}{1 + v^2}dv - \frac{1}{2}\frac{2v}{1 + v^2}dv &= \frac{dx}{x}
\end{aligned}$$

On integration we have

$$\begin{aligned}
\tan^{-1} v - \frac{1}{2}\log(1+v^2) &= \log x + c \\
2\tan^{-1} v &= \log(1+v^2) + \log x^2 + c \\
2\tan^{-1}\left(\frac{y}{x}\right) &= \log\left(1+\frac{y^2}{x^2}\right)(x^2) + c \\
2\tan^{-1}\left(\frac{y}{x}\right) &= \log\left(x^2+y^2\right) + c
\end{aligned}$$

Exercise:

1. Find all real-valued solutions of the following equations:

(a) $y' = \dfrac{y^2}{xy+x^2}$

(b) $y' = \dfrac{x^2+xy+y^2}{x^2}$

Non-homogeneous equation

Consider the equation of the form

$$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are constants and $c_1, c_2 \neq 0$ can be reduced to homogeneous equation.

Case 1:

If $a_1b_2 = a_2b_1$ then the substitution $a_1x + b_1y = v$ or $a_2x + b_2y = v$ reduces the given equation to one in which the variables are separated.

Case 2:

If $a_1b_2 \neq a_2b_1$ then the substitution $x = X + h$ and $y = Y + k$ where h and k are such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$ reduces the given equation to a homogeneous equation in X and Y. The final solution is got by replacing x and Y by $x - h$ and $y - k$ respectively.

Example 5.11. Consider the equation $\dfrac{dy}{dx} = \dfrac{x-y+1}{x+y-3}$

Here $a_1 = 1,\ b_1 = -1,\ c_1 = 1,\ a_2 = 1,\ b_2 = 1,\ c_2 = -3.$ Also $a_1b_2 = 1$ and $a_2b_1 = -1$. Hence $a_1b_2 \neq a_2b_1$.

Put $x = X + h$ and $y = Y + k$. Then $dx = dX$ and $dy = dY$.

Therefore $\dfrac{dy}{dx} = \dfrac{dY}{dX}$.

Then $\dfrac{dY}{dX} = \dfrac{X+h-Y-k+1}{X+h+Y+k-3}$

Choose h, k such that $h-k+1=0$ and $h+k-3=0$. Solving these two equations we have $h=1$ and $k=2$.

Therefore $\dfrac{dY}{dX} = \dfrac{X-Y}{X+Y}$. This is a homogeneous equation in X and Y.

Put $Y = vX$. Then $\dfrac{dY}{dX} = v + x\dfrac{dv}{dX}$.

$$\begin{aligned}
v + X\frac{dv}{dX} &= \frac{X-vX}{X+vX} \\
v + X\frac{dv}{dX} &= \frac{1-v}{1+v} \\
x\frac{dv}{dX} &= \frac{1-v}{1+v} - v \\
x\frac{dv}{dX} &= \frac{1-2v-v^2}{1+v} \\
\frac{1+v}{1-2v-v^2}dv &= \frac{dX}{X} \\
-\frac{1}{2}\frac{-2(1+v)}{1-2v-v^2}dv &= \frac{dX}{X}
\end{aligned}$$

On integration we have

$$\begin{aligned}
-\frac{1}{2}\log(1-2v-v^2) &= \log x + \log c_1 \\
(1-2v-v^2)^{-1/2} &= c_1 x \\
\frac{1}{(1-2v-v^2)^{1/2}} &= c_1 x \\
\frac{1}{(1-2v-v^2)} &= c_1^2 x^2 \\
(1-2v-v^2)x^2 &= \frac{1}{c_1^2} = c \\
(1-2\frac{y}{x} - \frac{y^2}{x^2})x^2 &= c \\
(x^2-2xy-y^2) &= c
\end{aligned}$$

Also we have $x = X + h = X + 1$ and $y = Y + k = Y + 2$.

Then $X = x - 1$ and $Y = y - 2$.

Therefore $(x-1)^2 - 2(x-1)(y-2) - (y-2)^2 = c$

$x^2 + 1 - 2x - 2xy + 4x + 2y - 4 - y^2 - 4 + 4y = c$

$x^2 - 2xy - y^2 + 2x + 6y = c + 8 = c_2.$

Example 5.12. Consider the equation

$$\frac{dy}{dx} = \frac{6x - 4y + 3}{3x - 2y + 1}$$

Here $a_1 = 6,\ b_1 = -4,\ c_1 = 3,\ a_2 = 3,\ b_2 = -2,\ c_2 = 1.$ Also $a_1 b_2 = -12$ and $a_2 b_1 = -12$. Hence $a_1 b_2 = a_2 b_1$.

Then substitute $3x - 2y = v$. Also on differentiation we have $3 - 2\frac{dy}{dx} = \frac{dv}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2(3x-2y)+3}{(3x-2y)+1} \\ \frac{dy}{dx} &= \frac{2v+3}{v+1} \end{aligned}$$

Therefore

$$\begin{aligned} 3 - 2\left(\frac{2v+3}{v+1}\right) &= \frac{dv}{dx} \\ \frac{dv}{dx} &= \frac{3v+3-4v-6}{v+1} \\ \frac{dv}{dx} &= \frac{-(v+3)}{v+1} \\ -\frac{v+1}{v+3}dv &= dx \end{aligned}$$

Now, $\frac{v+1}{v+3} = \frac{v+3-2}{v+3} = \frac{v+3}{v+3} - \frac{2}{v+3} = 1 - \frac{2}{v+3}$

$$-\left(1 - \frac{2}{v+3}\right) dv = dx$$

On integration we have

$$\begin{aligned} -v + 2\log(v+3) &= x + c \\ -(3x-2y) + 2\log(3x-2y+3) &= x + c \\ 2\log(3x-2y+3) &= 4x - 2y + c \\ \log(3x-2y+3) &= 2x - y + c_1 \end{aligned}$$

Exercise:

1. Find the solution of the following equations:

(a) $\frac{dy}{dx} = \frac{x+2y+3}{2x+y+3}$ (b) $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$

(c) $\frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3}$ (d) $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$

5.4 Exact equations

Suppose the first order equation $y' = f(x, y)$ is written in the form

$$y' = -\frac{M(x,y)}{N(x,y)}$$

Then

$$M(x, y) + N(x, y)y' = 0 \tag{5.8}$$

where M and N are real-valued function defined for real x and y on some rectangle R.

Definition 5.13. The equation $M(x, y) + N(x, y)y' = 0$ is said to be exact in R if there exists a function F having continuous first partial derivatives such that

$$\frac{\partial F}{\partial x} = M \text{ and } \frac{\partial F}{\partial y} = N \text{ in } R \tag{5.9}$$

Example 5.14. Consider the equation $ydx + xdy = 0$. Here $M = y$ and $N = x$.

Then there exists a function $F = xy$ such that $\frac{\partial F}{\partial x} = y = M$ and $\frac{\partial F}{\partial y} = x = N$. Hence the given equation is exact.

Theorem 5.15. *8uppose the equation*

$$M(x, y) + N(x, y)y' = 0 \tag{5.10}$$

is exact in a rectangle R and F is a real-valued function such that

$$\frac{\partial F}{\partial x} = M \text{ and } \frac{\partial F}{\partial y} = N \text{ in } R \tag{5.11}$$

Every differentiable function ϕ defined implicitly by a relation

$$F(x, y) = c \quad (c = constant)$$

is a solution of (5.10) whose graph lies in R arises this way.

Proof. Suppose $M(x,y)+N(x,y)y'=0$ is exact in R and $\dfrac{\partial F}{\partial x}=M$ and $\frac{\partial F}{\partial y}=N$ in R.

Then $\dfrac{\partial F(x,y)}{\partial x}+\dfrac{\partial F(x,y)}{\partial y}y'=0.$

If ϕ is any solution on some interval I, then
$\dfrac{\partial F(x,\phi(x))}{\partial x}+\dfrac{\partial F(x,\phi(x))}{\partial y}\phi'(x)=0$ for all $x\in I$.

If $\Phi(x)=F(x,\phi(x))$ then
$$\Phi'(x)=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\phi'(x)=0$$

That is $\Phi'(x)=0$ and so $\Phi(x)=c$, a constant. Hence $F(x,\phi(x))=c$.

Thus the solution ϕ must be a function given by $F(x,y)=c$.

Conversely, if ϕ is a differentiable function on some interval I defined by the relation $F(x,y)=c$. Then $F(x,\phi(x))=c$ for all $x\in I$.

Differentiating this we get
$$\frac{\partial F(x,\phi(x))}{\partial x}+\frac{\partial F(x,\phi(x))}{\partial y}\phi'(x)=0$$

Thus $M(x,\phi(x))+N(x,\phi(x))\phi'(x)=0$ Hence $\phi(x)$ is a solution of (5.10). □

Remark 5.16. If $M(x,y)+N(x,y)y'=0$ is exact then

$$\begin{aligned}M(x,y)dx+N(x,y)dy &= 0\\ \frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy &= 0\\ dF &= 0\end{aligned}$$

Example 5.17. Consider the equation $y'=-\frac{x}{y}$. Then

$$\begin{aligned}\frac{dy}{dx} &= -\frac{x}{y}\\ xdx+ydy &= 0\\ d\left(\frac{x^2+y^2}{2}\right) &= 0\\ \frac{x^2+y^2}{2} &= c_1\\ x^2+y^2 &= 2c_1=c\end{aligned}$$

Thus any differentiable function defined by the relation $x^2 + y^2 = c$, where c is a constant is a solution of given equation.

Note: Consider the equation with variable separated. Any such equation is a special case of an exact equation.

If we write equation $M(x, y) + N(x, y)y' = 0$ as

$$\begin{aligned} g(x)dx &= h(y)dy \\ \int g(x)dx &= \int h(y)dy \\ G(x) &= H(y) \end{aligned}$$

where $G(x) = \int g(x)dx$ and $H(y) = \int h(y)dy$. That is $G'(x) = g(x)$ and $H'(y) = h(y)$. It is clear that F is given by $F(x, y) = G(x) - H(y)$.

Theorem 5.18. *Let M, N be two real-valued functions which have continuous first partial derivatives on some rectangle*

$$R: \qquad |x - x_0| \leq a, \quad |y - y_0| \leq b.$$

Then the equation

$$M(x, y) + N(x, y)y' = 0$$

is exact in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{in } R \tag{5.12}$$

Proof. Suppose $M(x, y) + N(x, y)y' = 0$ is exact in R. Let F be a function which has continuous second derivatives such that

$$\frac{\partial F}{\partial x} = M \text{ and } \frac{\partial F}{\partial y} = N.$$

Then

$$\begin{aligned} \frac{\partial^2 F}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial M}{\partial y} \\ \frac{\partial^2 F}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial N}{\partial x} \end{aligned}$$

Since $\dfrac{\partial^2 F}{\partial y \partial x} = \dfrac{\partial^2 F}{\partial x \partial y}$ for a function F we have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

conversely, suppose $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. We need to find a function F satisfying $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$.

Suppose if we had such a function then

$$\begin{aligned} F(x,y) - F(x_0,y_0) &= F(x,y) - F(x_0,y) + F(x_0,y) - F(x_0,y_0) \\ &= \int_{x_0}^{x} \frac{\partial F(s,y)}{\partial x}\, ds + \int_{y_0}^{y} \frac{\partial F(x_0,t)}{\partial y}\, dt \\ &= \int_{x_0}^{x} M(s,y)\, ds + \int_{y_0}^{y} N(x_0,t)\, dt \end{aligned} \tag{5.13}$$

Similarly we have

$$\begin{aligned} F(x,y) - F(x_0,y_0) &= F(x,y) - F(x,y_0) + F(x,y_0) - F(x_0,y_0) \\ &= \int_{y_0}^{y} \frac{\partial F(x,t)}{\partial y}\, dt + \int_{x_0}^{x} \frac{\partial F(s,y_0)}{\partial x}\, ds \\ &= \int_{y_0}^{y} N(x,t)\, dt + \int_{x_0}^{x} M(s,y_0)\, ds \end{aligned} \tag{5.14}$$

Now we define F by

$$F(x,y) = \int_{x_0}^{x} M(s,y)\, ds + \int_{y_0}^{y} N(x_0,t)\, dt \tag{5.15}$$

From (5.15) we have $F(x_0,y_0) = 0$. Also

$$\frac{\partial F(x,y)}{\partial x} = \frac{\partial}{\partial x}\left[\int_{x_0}^{x} M(s,y)\, ds + \int_{y_0}^{y} N(x_0,t)\, dt\right] = M(x,y)$$

for all (x,y) in R.

From (5.14) we would guess that F is also given by

$$F(x,y) = \int_{x_0}^{x} M(s,y_0)\, ds + \int_{y_0}^{y} N(x,t)\, dt \tag{5.16}$$

Therefore $\dfrac{\partial F(x,y)}{\partial y} = \dfrac{\partial}{\partial y}\left[\int\limits_{x_0}^{x} M(s,y_0)\ ds + \int\limits_{y_0}^{y} N(x,t)\ dt\right] = N(x,y)$

for all (x,y) in R. Thus we found our F.

In order to show that (5.16) is valid where F is the function given by (5.15).

We consider the difference

$$
\begin{aligned}
&F(x,y) - \left[\int_{x_0}^{x} M(s,y_0)\ ds + \int_{y_0}^{y} N(x,t)\ dt\right] \\
&= \left[\int\limits_{x_0}^{x} M(s,y)\ ds + \int\limits_{y_0}^{y} N(x_0,t)\ dt\right] - \left[\int\limits_{x_0}^{x} M(s,y_0)\ ds + \int\limits_{y_0}^{y} N(x,t)\ dt\right] \\
&= \left[\int\limits_{x_0}^{x} M(s,y)\ ds - \int\limits_{x_0}^{x} M(s,y_0)\ ds\right] - \left[\int\limits_{y_0}^{y} N(x,t)\ dt - \int\limits_{y_0}^{y} N(x_0,t)\ dt\right] \\
&= \int\limits_{x_0}^{x} (M(s,y) - M(s,y_0))\ ds - \int\limits_{y_0}^{y} (N(x,t) - N(x_0,t))\ dt \\
&= \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} \frac{\partial M(s,t)}{\partial t}\ dt\ ds - \int\limits_{y_0}^{y}\int\limits_{x_0}^{x} \frac{\partial N(s,t)}{\partial s}\ ds\ dt \\
&= \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} \left(\frac{\partial M(s,t)}{\partial t} - \frac{\partial N(s,t)}{\partial s}\right)\ dt\ ds \\
&= 0, \qquad \text{since } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
\end{aligned}
$$

Therefore $F(x,y) = \int_{x_0}^{x} M(s,y_0)\ ds + \int_{y_0}^{y} N(x,t)\ dt$. This completes the proof. □

Rules to find the solution of exact equations

1. Verify whether given equation $M(x,y) + N(x,y)y' = 0$ is exact.
2. If exact, integrate M with respect to x keeping y as constant.
3. Find out those terms in N which are free from x and integrate those terms with respect to y.

4. The sum of these two expressions equated to an arbitrary constant is the required general solution of given exact equation.

Example 5.19. Consider the equation

$$y' = \frac{3x^2 - 2xy}{x^2 - 2y}$$

This equation can be written as $(3x^2 - 2xy)dx - (x^2 - 2y)dy = 0$. That is $(3x^2 - 2xy)dx + (2y - x^2)dy = 0$.

Here $M = 3x^2 - 2xy$ and $N = 2y - x^2$. Then $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = -2x$. Hence the given equation is exact.

Then integrating M with respect to x keepin y as constant we have

$$\begin{aligned} \int M\ dx &= \int (3x^2 - 2xy)\ dx \\ &= \frac{3x^3}{3} - \frac{2x^2y}{2} \\ &= x^3 - x^2y \end{aligned}$$

and $S =$ the terms of N free from $x = 2y$. Then
$\displaystyle\int S\ dy = \int 2y\ dy = y^2.$

Hence $x^3 - x^2y + y^2 = c$ is the general solution of given equation.

Exercise:
Verify the following equations are exact and solve them.
(a) $2xy\ dx + (x^2 + 3y^2)\ dy = 0$
(b) $x^2y^3\ dx + x^3y^2\ dy = 0$
(c) $(x + y)\ dx + (x - y)\ dy = 0$

Integrating factor

Consider the equation

$$M(x, y)dx + N(x, y)dy = 0. \tag{5.17}$$

Sometimes the equation (5.17) may not be exact. So we find a function $'u'$ nowhere zero such that

$$u(x, y)M(x, y)\ dx + u(x, y)N(x, y)\ dy = 0$$

is exact. Such a function $'u'$ is called an integrating factor.

Example 5.20. Consider the equation $y\ dx - x\ dy = 0$.

Here $M = y$ and $N = -x$. Then $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -1$. Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and so the given equation is not exact.

When the equation is multiplied by $\frac{1}{y^2}$ we get

$$\frac{y\ dx - x\ dy}{y^2} = 0$$

$\Rightarrow d\left(\frac{x}{y}\right) = 0$

$\Rightarrow \frac{x}{y} = c_1$

$\Rightarrow y = \frac{x}{c_1} = cx$ where $\frac{1}{c_1} = c$

That is $y = cx$. Thus the equation becomes exact. Hence the integrating factor is $\frac{1}{y^2}$.

Result:

1. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x)$, a function of x only then $\mu = e^{\int g(x)\ dx}$ is an integrating factor of $M\ dx + N\ dy = 0$.

2. If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = h(y)$, a function of y only then $\mu = e^{\int h(y)\ dy}$ is an integrating factor of $M\ dx + N\ dy = 0$.

3. If $M\ dx + N\ dy = 0$ is a homogeneous equation where M and N are homogeneous function of degree n and if $Mx + Ny \neq 0$, then $\frac{1}{mx + ny}$ is an integrating factor.

4. If $M\ dx + N\ dy = 0$ is of the form $yf(xy)dx + xg(xy)dy = 0$ where $f(xy) \neq g(xy)$ then $\frac{1}{mx - ny}$ is an integrating factor.

Example 5.21. Consider the equation $(x^2 + y^2 + x)dx + xydy = 0$.

Here $\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = y$. Clearly the given equation is not exact.

Then $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{y}{xy} = \frac{1}{x}$.

Therefore the integrating factor is $\mu = e^{\int g(x)\ dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$.

Exercise:
Find an integrating factor for the following equation.
(a) $(2y^2+2)dx + 3xy^2dy = 0$
(b) $(5x^2y^2+2y)dx + (3x^4y+2x)dy = 0$

5.5 Method of successive approximations

Consider the equation

$$y' = f(x,y) \tag{5.18}$$

where f is any continuous real-valued function defined on some rectangle

$$R: \quad |x-x_0| \leq a, \quad |y-y_0| \leq b, \quad (a,b>0),$$

in the real $(x,y)-$plane.

To show: On some interval I containing x_0 there is a solution ϕ of (5.18) satisfying

$$\phi(x_0) = y_0 \tag{5.19}$$

That is there is a real-valued differentiable function ϕ satisfying $\phi(x_0) = y_0$ such that the points $(x, \phi(x))$ are in R for x in I, and $\phi'(x) = f(x, \phi(x))$ for all x in I. Such a function ϕ is called a solution to the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0 \quad \text{on } I. \tag{5.20}$$

We now show that the initial value problem is equivalent to an integral equation namely

$$y = y_0 + \int_{x_0}^{x} f(t,y)\, dt \qquad \text{on} \;\; I. \tag{5.21}$$

Suppose ϕ is a solution to (5.21) on I with $(x, \phi(x))$ is in R, then

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$$

where ϕ is a real valued continuous function on I.

Theorem 5.22. *A function ϕ is a solution of the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$ on an interval I if and only if it is a solution of the integral equation $y = y_0 + \int_{x_0}^{x} f(t,y)\, dt$ on I.*

Proof. Suppose ϕ is a solution of the initial value problem $y' = f(x, y), \quad y(x_0) = y_0$ on I. Then

$$\phi'(t) = f(t, \phi(t)) \tag{5.22}$$

on I. Since ϕ is continuous on I anf f is continuous on R, the function F defined by $F(t) = f(t, \phi(t))$ is continuous on I. Integrating (5.22) from x_0 to x we obtain

$$\begin{aligned}
\int_{x_0}^{x} \phi'(t)\, dt &= \int_{x_0}^{x} f(t, \phi(t))\, dt \\
[\phi(t)]_{x_0}^{x} &= \int_{x_0}^{x} f(t, \phi(t))\, dt \\
\phi(x) - \phi(x_0) &= \int_{x_0}^{x} f(t, \phi(t))\, dt \\
\phi(x) &= \phi(x_0) + \int_{x_0}^{x} f(t, \phi(t))\, dt
\end{aligned}$$

and since $\phi(x_0) = y_0$, we have

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$$

Hence ϕ is a solution of $y = y_0 + \int_{x_0}^{x} f(t, y)\, dt$

Conversely, suppose ϕ is a solution of $y = y_0 + \int_{x_0}^{x} f(t, y)\, dt$ on I. Then

$$\phi(x) = \phi(x_0) + \int_{x_0}^{x} f(t, \phi(t))\, dt \tag{5.23}$$

On differentiating and using fundamental theorem of calculus, we have

$$\phi'(x) = \frac{d}{dx} \int_{x_0}^{x} f(t, \phi(t))\, dt = f(x, \phi(x)).$$

Also from (5.23), we have $\phi(x_0) = y_0$.

Hence ϕ is a solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$. Hence the theorem. □

Successive approximation

Consider the function ϕ_0 defined by

$$\phi_0(x) = y_0$$

This function satisfies the initial condition $\phi_0(x_0) = y_0$, but does not in general satisfy (5.21). However, if we compute

$$\begin{aligned}\phi_1(x) &= y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\ dt \\ &= y_0 + \int_{x_0}^{x} f(t, y_0)\ dt\end{aligned}$$

We might expect that ϕ_1 is a closer approximation to a solution than ϕ_0. If we continue the process and define successively $\phi_0(x) = y_0$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\ dt, \qquad (k = 0, 1, 2, \cdots) \tag{5.24}$$

on taking the limit as $k \to \infty$, that we would obtain

$$\phi_k(x) \to \phi(x)$$

where ϕ would satisfy

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\ dt$$

This ϕ is our desired solution. We call the functions $\phi_1, \phi_2, \cdots$ defined by (5.24) successive approximations to a solution of the integral equation (5.21) or the initial value problem (5.20).

Example 5.23. Consider the initial value problem $y' = xy, \quad y(0) = 1$.

Successive approximation method

Then the integral equation corresponding to this initial value problem is

$$y = 1 + \int_{0}^{x} ty\ dt$$

and the successive approximation are given by $\phi_0(x) = 1$

$$\phi_{k+1}(x) = 1 + \int_{0}^{x} t\phi_k(t)\ dt$$

.

Thus

$$\begin{aligned}
\phi_1(x) &= 1+\int_0^x t\phi_0(t)\ dt \\
&= 1+\int_0^x t\ dt \\
&= 1+\left(\frac{t^2}{2}\right)_0^x \\
&= 1+\frac{x^2}{2}
\end{aligned}$$

$$\begin{aligned}
\phi_2(x) &= 1+\int_0^x t\phi_1(t)\ dt \\
&= 1+\int_0^x t\left(1+\frac{t^2}{2}\right)\ dt \\
&= 1+\left(t+\frac{t^3}{2}\right)\ dt \\
&= 1+\left(\frac{t^2}{2}+\frac{t^4}{(4)(2)}\right)_0^x \\
&= 1+\left(\frac{x^2}{2}+\frac{x^4}{2^2\ 2\ !}\right) \\
&= 1+\frac{x^2}{2}+\frac{1}{2\ !}\left(\frac{x^2}{2}\right)^2
\end{aligned}$$

Then by induction we obtain

$$\phi_k(x) = 1+\frac{x^2}{2}+\frac{1}{2\ !}\left(\frac{x^2}{2}\right)^2+\cdots+\frac{1}{k\ !}\left(\frac{x^2}{2}\right)^k$$

It is clear that $\phi_k(x)$ is the partial sum for the series expansion of the function $\phi(x) = e^{x^2/2}$. Hence as limit $k\to\infty$, $\phi_k(x)\to\phi(x)$ for all x.

Thus $\phi(x) = e^{x^2/2}$ is the solution of given equation.

Usual method consider the equation $y' = xy$. Then by variable separable method we have

$$
\begin{aligned}
\frac{dy}{dx} &= xy \\
\frac{dy}{y} &= x\ dx \\
\int \frac{dy}{y} &= \int x\ dx \\
\log y &= \frac{x^2}{2} + c \\
y &= e^{(x^2/2)+c} \\
y(0) = 1 &\Rightarrow c = 0
\end{aligned}
$$

Hence $y = e^{x^2/2}$ is the solution of the given equation.

Note:

Since f is continuous on R, it is bounded there. Hence there exists $M > 0$ such that $|f(x, y)| \leq M$ for all $(x, y) \in R^*$.

Theorem 5.24. *The successive approximations ϕ_k, defined by (5.24), exist as continuous functions on*

$$I: \quad |x - x_0| \leq \alpha = \textit{minimum } \{a, b/M\},$$

and $(x, \phi(x))$ is in R for x in I. Indeed, the ϕ_k satisfy

$$|\phi_k(x) - y_0| \leq M|x - x_0| \tag{5.25}$$

for all x in I.

Proof. **Note:** Since for $x \in I$, $|x - x_0| \leq b/M$, the inequality $|\phi_k(x) - y_0| \leq M|x - x_0| \leq b$ for all x in I, which shows that $(x, \phi_k(x))$ are in R for x in I.

The geometric interpretation of the inequality $|\phi_k(x) - y_0| \leq M|x - x_0|$ is that the graph of each ϕ_k lies in the region T in R bounded by the two lines

$$y - y_0 = M(x - x_0) \text{ and } y - y_0 = -M(x - x_0)$$

and the lines

$$x - x_0 = \alpha \text{ and } x - x_0 = -\alpha$$

Proof of theorem

We prove this by induction.

Clearly ϕ_0 exists on I as a continuous function, and satisfies (5.25) with $k = 0$, since $|\phi_0(x) - y_0| = |y_0 - y_0| = 0 \le M|x - x_0|$.

Also, since (x, y_0) is in R, $(x, \phi_0(x))$ is in R. Now,

$$
\begin{aligned}
|\phi_1(x) - y_0| &= \left| y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\ dt - y_0 \right| \\
&= \left| \int_{x_0}^{x} f(t, \phi_0(t))\ dt \right| \\
&\le \left| \int_{x_0}^{x} |f(t, \phi_0(t))|\ dt \right| \\
&\le \left| M \int_{x_0}^{x} dt \right| \\
&= M|x - x_0|
\end{aligned}
$$

Therefore $|\phi_1(x) - y_0| \le M|x - x_0|$. Thus ϕ_1 satisfies the inequality. Since f is continuous on R, the function F_0 defined by $F_0 = f(t, y_0)$ is continuous on I.

Thus ϕ_1 given by

$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\ dt = y_0 + \int_{x_0}^{x} f(t, y_0)\ dt = y_0 + \int_{x_0}^{x} F_0(t)\ dt$ is continuous on I.

Suppose we assume that the theorem is true for the functions $\phi_0, \phi_1, \phi_2, \cdots \phi_k$.

To prove: The result is true for ϕ_{k+1}.

We know that $(t, \phi_k(t))$ is in R for $t \in I$. Thus the function F_k given by $F_k(t) = f(t, \phi_k(t))$ exist for $t \in I$. It is continuous on I, since f is continuous on R and ϕ_k is continuous on I.

Therefore ϕ_{k+1} is given by

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F_k(t)\ dt$$

exists as a continuous function on I.

Also,

$$
\begin{aligned}
|\phi_{k+1}(x) - y_0| &\leq \left| \int_{x_0}^{x} F_k(t)\ dt \right| \\
&\leq \left| \int_{x_0}^{x} |F_k(t)|\ dt \right| \\
&\leq M|x - x_0|
\end{aligned}
$$

Thus $|\phi_{k+1}(x) - y_0| \leq M|x - x_0|$. Hence ϕ_{k+1} satisfies the inequality. Hence the theorem. □

Example 5.25. Consider the initial value problem $y' = 3y + 1, \quad y(0) = 2$.

(a) Compute the first four approximations $\phi_1, \phi_1, \phi_2, \phi_3$ to the solution.

(b) Compute the solution by using one of the methods in section (5.2) to (5.6)

(c) Compare the results of (b) and (c).

Solution:
(a) Given $y' = 3y + 1, \quad y(0) = 2$.

Here $f(x, y) = 3y + 1$, $x_0 = 0,\ y_0 = 2$. Then The integral equation corresponding to initial value problem is

$$y = y_0 + \int_{x_0}^{x} f(t, y)dt = 2 + \int_{0}^{x} (3y + 1)\ dt$$

and successive approximation are given by

$$\phi_0(x) = y_0 = 2$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\ dt, \qquad (k = 0, 1, 2, \cdots)$$

That is $\phi_{k+1}(x) = 2 + \int_{0}^{x} (3\phi_k(t) + 1)\ dt$.

Thus

$$\begin{aligned}
\phi_1(x) &= 2+\int_0^x (3\phi_0(t)+1)\ dt \\
&= 2+\int_0^x (6+1)\ dt \\
\phi_1(x) &= 2+7x \\
\phi_2(x) &= 2+\int_0^x (3\phi_1(t)+1)\ dt \\
&= 2+\int_0^x (3(2+7t)+1)\ dt \\
&= 2+\int_0^x (7+21t)\ dt \\
\phi_2(x) &= 2+7x+\frac{21x^2}{2} \\
\phi_3(x) &= 2+\int_0^x (3\phi_2(t)+1)\ dt \\
&= 2+\int_0^x (3(2+7t+\frac{21t^2}{(}2))+1)\ dt \\
&= 2+\int_0^x (7+21t+\frac{63t^2}{2})\ dt \\
\phi_2(x) &= 2+7x+\frac{21x^2}{2}+\frac{63x^3}{6}
\end{aligned}$$

(b) Given $y' = 3y+1,\ y(0) = 2$. This is a linear equation of the form $y' + g(x)y = h(x)$. Here $g(x) = -3$ and $h(x = 1$. Clearly g, h are continuous functions. Then the solution is given by

$$\phi(x) = e^{-Q(x)} \int_{x_0}^{x} e^{Q(t)} h(t)dt + ce^{-Q(x)}, \text{ where } Q(x) = \int_{x_0}^{x} g(t)dt$$

Therefore $Q(x) = \int_0^x (-3)dt = -3x$. Thus

$$\phi(x) = e^{3x}\int_0^x e^{-3t}dt + ce^{3x} = \frac{-1}{3} + \frac{e^{3x}}{3} + ce^{3x}.$$

Also $y(0) = 2$ implies that $c = 2$. Hence $\phi(x) = \frac{1}{3}\left(7e^{3x} - 1\right)$.

(c) Now using the series expansion of e^{3x}, we have

$$e^{3x} = 1 + \frac{3x}{1\,!} + \frac{(3x)^2}{2\,!} + \frac{(3x)^3}{3\,!} + \cdots$$

$$\begin{aligned}
\phi(x) &= \frac{1}{3}\left(7e^{3x} - 1\right) \\
&= \frac{1}{3}\left[7\left(1 + \frac{3x}{1\,!} + \frac{(3x)^2}{2\,!} + \frac{(3x)^3}{3\,!} + \cdots\right) - 1\right] \\
&= \frac{1}{3}\left[6 + 21x + \frac{63x^2}{2\,!} + \frac{7(3x)^3}{3\,!} + \cdots\right] \\
&= 2 + 7x + \frac{21x^2}{2\,!} + \frac{63x^3}{3\,!} + \cdots
\end{aligned}$$

Taking limit as $k \to \infty$ we have $\phi_k(x) \to \phi(x)$. Hence $\phi(x) = \frac{1}{3}\left(7e^{3x} - 1\right)$ is the solution of given initial value problem.

Exercise:

Compute the first four approximation $\phi_0, \phi_1, \phi_2, \phi_3$.

(a) $y' = x^2 + y^2,\ y(0) = 0$ (b) $y' = 1 + xy,\ y(0) = 1$

(c) $y' = y^2,\ y(0) = 0$ (d) $y' = y^2,\ y(0) = 1$

5.6 Lipschitz condition

Let f be a function defined for (x, y) in a set S. We say f satisfy a Lipschitz condition on S if there exists a constant $K > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \le K|y_1 - y_2|$$

for all $(x, y_1), (x, y_2)$ in S. The constant K is called a Lipschitz constant.

Note:

If f is continuous and satisfies a Lipschitz condition on the rectangle R, then the successive approximations converge to a solution of the initial value problem on $|x - x_0| \le \alpha$.

Theorem 5.26. *Suppose S is either a rectangle*

$$|x - x_0| \leq a, \qquad |y - y_0| \leq b, \quad (a, b > 0),$$

or a strip

$$|x - x_0| \leq a, \qquad |y| < \infty, \quad (a > 0),$$

and that f is a real-valued function defined on S such that $\dfrac{\partial f}{\partial y}$ exists, is continuous on S, and

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \leq K, \qquad ((x, y) \text{ in } S),$$

for some $K > 0$. Then f satisfies a Lipschitz condition on S with Lipschitz constant K.

Proof. Supppose $\left|\dfrac{\partial f}{\partial y}(x, y)\right| \leq K, \quad ((x, y) \text{ in } S)$. Then we have

$$\begin{aligned} f(x, y_1) - f(x, y_2) &= \int_{y_2}^{y_1} \frac{\partial f}{\partial y}(x, t)\, dt \\ |f(x, y_1) - f(x, y_2)| &= \left|\int_{y_2}^{y_1} \frac{\partial f}{\partial y}(x, t)\, dt\right| \\ &\leq \left|\int_{y_2}^{y_1} \left|\frac{\partial f}{\partial y}(x, t)\right|\, dt\right| \\ &\leq \left|\int_{y_2}^{y_1} K\, dt\right| \\ &= K|y_1 - y_2| \end{aligned}$$

Therefore $|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$ for all $(x, y_1), (x, y_2)$ in S. Hence f satisfies Lipschitz condition on S. □

Example 5.27. Consider a function $f(x, y) = xy^2$ in $R: \ |x| \leq 1, \ \ |y| \leq 1$.

Now for (x, y) in R,

$$\begin{aligned} \frac{\partial f}{\partial y}(x, y) &= 2xy \\ \left|\frac{\partial f}{\partial y}(x, y)\right| &= |2xy| \end{aligned}$$

$$
\begin{aligned}
&\leq 2\,|x|\,|y| \\
&\leq 2, \quad \text{since} \quad |x| \leq 1, \quad |y| \leq 1
\end{aligned}
$$

Alternate method:

Now for $(x, y_1), (x, y_2)$ in R,

$$
\begin{aligned}
|f(x, y_1) - f(x, y_2)| &= |xy_1^2 - xy_2^2| \\
&= |x|\,|y_1^2 - y_2^2| \\
&= |x|\,|(y_1 + y_2)\,(y_1 - y_2)| \\
&= |x|\,|y_1 + y_2|\,|y_1 - y_2| \\
&\leq |x|\,(|y_1| + |y_2|)\,|y_1 - y_2| \\
&\leq 2|y_1 - y_2|, \quad \text{since} \quad |x| \leq 1, \quad |y| \leq 1
\end{aligned}
$$

Thus $|f(x, y_1) - f(x, y_2)| \leq 2|y_1 - y_2|$. Hence f satisfies Lipschitz condition on R.

Example 5.28. Consider a function $f(x, y) = xy^2$ on the strip $S: \ |x| \leq 1, \quad |y| < \infty$.

Now for (x, y) in R,

$$
\begin{aligned}
\frac{\partial f}{\partial y}(x, y) &= 2xy \\
\left|\frac{\partial f}{\partial y}(x, y)\right| &= |2xy| \\
&\leq 2\,|x|\,|y| \\
&< \infty, \quad \text{since} \quad |x| \leq 1, \quad |y| < \infty
\end{aligned}
$$

Hence f does not satisfy Lipschitz condition on the strip.

Example 5.29. Consider a continuous function $f(x, y) = y^{2/3}$ on the rectangle $R: \ |x| \leq 1, \quad |y| \leq 1$. Now for (x, y) in R,

$$
\begin{aligned}
\frac{\partial f}{\partial y}(x, y) &= \frac{2}{3}\,y^{-1/3} \\
&\leq \frac{2}{3}|y^{-1/3}| \\
&\to \infty, \quad \text{as } y \to 0
\end{aligned}
$$

Hence f does not satisfy Lipschitz condition on the strip.

Exercise:
Show that the following function satisfies Lipschitz condition on the set S.
(a) $f(x,y) = 4x^2 + y^2$ on $S : |x| \leq 1, \quad |y| \leq 1$
(b) $f(x,y) = x^2 \cos^2 y + y \sin^2 x$ on $S : |x| \leq 1, \quad |y| < \infty$.

5.7 Convergence of Successive approximation

We now prove the main existence theorem.

Theorem 5.30. (*Existence Theorem*) *Let f be a continuous real-valued function on the rectangle*

$$R : \quad |x - x_0| \leq a, \quad |y - y_0| \leq b, \quad (a, b > 0),$$

and let $|f(x,y)| \leq M$ for all (x,y) in R. Further suppose that f satisfies a Lipschitz condition with constant K in R. Then the successive approximations

$$\phi_0(x) = y_0, \quad \phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\ dt, \quad (k = 0, 1, 2, \cdots),$$

converge on the interval

$$I : \quad |x - x_0| \leq \alpha = \text{ minimum } \{a, b/M\}$$

to a solution ϕ of the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0 \text{ on } I.$$

Proof. **(a) Convergence of** $\{\phi_k(x)\}$.
ϕ_k may be written as

$\phi_k = \phi_0 - \phi_0 + \phi_1 - \phi_1 + \cdots + \phi_{k-1} - \phi_{k-1} + \phi_k$
$\phi_k = \phi_0 + (\phi_1 - \phi_0) + (\phi_2 - \phi_1) + \cdots + (\phi_k - \phi_{k-1})$

Hence ϕ_k is a partial sum for the series

$$\phi_0(x) + \sum_{p=1}^{\infty} (\phi_p(x) - \phi_{p-1}(x)) \tag{5.26}$$

To show: The sequence $\{\phi_k(x)\}$ converges. It is equivalent to show that the series (5.26) converges. Then by Theorem 5.24, the function ϕ_p all exists as continuous function on I and $(x, \phi_p(x))$ is in R for x in I.

Moreover $|\phi_1(x) - \phi_0(x)| \leq M\ |x - x_0|$ for x in I. That is $|\phi_1(x) - y_0| \leq M\ |x - x_0|$

for x in I.

Now

$$\begin{aligned}
\phi_2(x) - \phi_1(x) &= \left(y_0 + \int_{x_0}^{x} f(t, \phi_1(t))\ dt \right) - \left(y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\ dt \right) \\
&= \int_{x_0}^{x} [f(t, \phi_1(t)) - f(t, \phi_0(t))]\ dt \\
|\phi_2(x) - \phi_1(x)| &= \left| \int_{x_0}^{x} [f(t, \phi_1(t)) - f(t, \phi_0(t))]\ dt \right| \\
&\leq \left| \int_{x_0}^{x} |f(t, \phi_1(t)) - f(t, \phi_0(t))|\ dt \right|
\end{aligned}$$

Since f satisfies Lipschitz condition, $|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$. Therefore,

$$\begin{aligned}
|\phi_2(x) - \phi_1(x)| &\leq \left| \int_{x_0}^{x} K\ |\phi_1(t) - \phi_0(t)|\ dt \right| \\
&\leq K\ M \left| \int_{x_0}^{x} |t - x_0|\ dt \right| \\
&= K\ M \left| \int_{x_0}^{x} (t - x_0)\ dt \right| \\
&= K\ M \left[\frac{(t - x_0)^2}{2} \right]_{x_0}^{x}, \quad \text{since} \ \ x - x_0 \geq 0 \\
&= K\ M\ \frac{(x - x_0)^2}{2}
\end{aligned}$$

Therefore $|\phi_2(x) - \phi_1(x)| \leq K\ M\ \dfrac{(x - x_0)^2}{2}$ if $x \geq x_0$.

If $x \leq x_0$, the same result is valid.

We shall prove by induction that

$$|\phi_p(x) - \phi_{p-1}(x)| \leq \frac{M\ K^{p-1}|x - x_0|^p}{p\ !} \ \ \text{for all } x \in I. \tag{5.27}$$

The result is true for $p = 1$ and $p = 2$.

Let us assume that $x \geq x_0$ (the proof is similar for $x \leq x_0$).

Assume the result (5.27) is true for $p = m$.

To prove: For $p = m + 1$.

$$\begin{aligned}
\phi_{m+1}(x) - \phi_m(x) &= \left(y_0 + \int_{x_0}^{x} f(t, \phi_m(t))\ dt \right) - \left(y_0 + \int_{x_0}^{x} f(t, \phi_{m-1}(t))\ dt \right) \\
&= \int_{x_0}^{x} [f(t, \phi_m(t)) - f(t, \phi_{m-1}(t))]\ dt \\
|\phi_{m+1}(x) - \phi_m(x)| &= \left| \int_{x_0}^{x} [f(t, \phi_m(t)) - f(t, \phi_{m-1}(t))]\ dt \right| \\
&\leq \left| \int_{x_0}^{x} |f(t, \phi_m(t)) - f(t, \phi_{m-1}(t))|\ dt \right| \\
&\leq \left| \int_{x_0}^{x} K\ |\phi_m(t) - \phi_{m-1}(t)|\ dt \right|, \quad \text{by Lipschitz condition} \\
&\leq K \left| \int_{x_0}^{x} \frac{M\ K^{m-1} |t - x_0|^m}{m\ !}\ dt \right| \\
&= \frac{M\ K^m}{m\ !} \left| \int_{x_0}^{x} |t - x_0|^m\ dt \right| \\
&= \frac{M\ K^m}{m\ !} \left| \int_{x_0}^{x} (t - x_0)^m\ dt \right|, \quad \text{since} \quad x - x_0 \geq 0 \\
&= \frac{M\ K^m}{m\ !} \left[\frac{(t - x_0)^{m+1}}{(m+1)} \right]_{x_0}^{x} \\
&= \frac{M\ K^m (x - x_0)^{(m+1)}}{(m+1)\ !}, \quad \text{for } x \geq x_0
\end{aligned}$$

Therefore $|\phi_{m+1}(x) - \phi_m(x)| \leq \dfrac{M\ K^m |x - x_0|^{(m+1)}}{(m+1)\ !}$ for all x in I.

Hence (5.27) is true for $p = m + 1$. Hence (5.27) is true for all p. The infinite series

$$\phi_0(x) + \sum_{p=1}^{\infty} (\phi_p(x) - \phi_{p-1}(x))$$

is absolutely convergent, that is, the series

$$|\phi_0(x)| + \sum_{p=1}^{\infty} |\phi_p(x) - \phi_{p-1}(x)| \tag{5.28}$$

is convergent on I. Indeed, from (5.27), we see that

$$|\phi_p(x) - \phi_{p-1}(x)| \le \frac{M\ K^p |x - x_0|^p}{K\ p\,!}$$

which shows that P^{th} term of the series in (5.28) is less than or equal to $\frac{M}{K}$ times P^{th} term of power series for $e^{K\ |x-x_0|}$.

Since the power series for $e^{K\ |x-x_0|}$ is convergent, the series (5.28) is convergent for all $x \in I$. Therefore (5.26) is convergent on I. Hence k^{th} partial sum of (5.26) which is just $\phi_k(x)$ tends to a limit $\phi(x)$ as $k \to \infty$ for each $x \in I$.

(b) Properties of the limit ϕ

This limit function ϕ is a solution to our problem on I.

To show ϕ is continuous on I.

If x_1, x_2 are in I.

$$\begin{aligned}
|\phi_{k+1}(x_1) - \phi_{k+1}(x_2)| &= \left| \int_{x_0}^{x_1} f(t, \phi_k(t))\ dt - \int_{x_0}^{x_2} f(t, \phi_k(t))\ dt \right| \\
&= \left| \int_{x_0}^{x_1} f(t, \phi_k(t))\ dt + \int_{x_2}^{x_0} f(t, \phi_k(t))\ dt \right| \\
&= \left| \int_{x_2}^{x_1} f(t, \phi_k(t))\ dt \right| \\
&\le \left| \int_{x_2}^{x_1} |f(t, \phi_k(t))|\ dt \right| \\
&\le M\ |x_1 - x_2|, \quad \text{since } |f(x,y)| \le M, \text{ for all } (x,y) \in R.
\end{aligned}$$

By letting $k \to \infty$, $\phi_k(x) \to \phi(x)$. Therefore

$$|\phi(x_1) - \phi(x_2)| \le M\ |x_1 - x_2| \tag{5.29}$$

This shows that as $x_2 \to x_1$, $\phi(x_2) \to \phi(x_1)$. That is ϕ is continuous on I.

Also letting $x_1 = x, \quad x_2 = x_0$ in (5.29), we obtain

$$|\phi(x) - \phi(x_0)| \leq M\ |x - x_0|$$

That is $|\phi(x) - y_0| \leq M\ |x - x_0|$ for x in I. Therefore the points $(x, \phi(x))$ are in R for all x in I.

(c) Estimate for $|\phi(x) - \phi_k(x)|$

We have $\phi(x) = \phi_0(x) + \sum_{p=1}^{\infty} (\phi_p(x) - \phi_{p-1}(x))$ and $\phi_k(x) = \phi_0(x) + \sum_{p=1}^{k} (\phi_p(x) - \phi_{p-1}(x))$.

Using (5.27) we find that

$$\begin{aligned}
|\phi(x) - \phi_k(x)| &= \left| \sum_{p=0}^{\infty} [\phi_p(x) - \phi_{p-1}(x)] - \sum_{p=0}^{k} [\phi_p(x) - \phi_{p-1}(x)] \right| \\
&= \left| \sum_{p=k+1}^{\infty} [\phi_p(x) - \phi_{p-1}(x)] \right| \\
&\leq \sum_{p=k+1}^{\infty} |\phi_p(x) - \phi_{p-1}(x)| \\
&\leq \sum_{p=k+1}^{\infty} \frac{M}{K} \frac{K^p |x - x_0|^p}{p\ !} \\
&= \sum_{p=k+1}^{\infty} \frac{M}{K} \frac{(Ka)^p}{p\ !} \quad \text{where} \quad |x - x_0| \leq a \\
&= \frac{M}{K} \sum_{p=k+1}^{\infty} \frac{(Ka)^p}{p\ !} \\
&\leq \frac{M}{K} \frac{(Ka)^{k+1}}{(k+1)\ !} \sum_{p=0}^{\infty} \frac{(Ka)^p}{p\ !} \\
&\leq \frac{M}{K} \frac{(Ka)^{k+1}}{(k+1)\ !}\ e^{Ka}
\end{aligned}$$

Therefore

$$|\phi(x) - \phi_k(x)| \leq \frac{M}{K} \frac{(Ka)^{k+1}}{(k+1)\ !}\ e^{Ka} \tag{5.30}$$

Letting $\epsilon_k = \dfrac{(Ka)^{k+1}}{(k+1)\ !}$. Since ϵ_k is a general term for the series for e^{Ka} we see that $\epsilon_k \to 0$ as $k \to \infty$. In terms of ϵ_k, (5.30) may be written as

$$|\phi(x) - \phi_k(x) \leq \frac{M}{K}\ e^{Ka}\ \epsilon_k.$$

(d) Limit ϕ is a solution

To show

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\ dt \quad \text{for all} \ \ x \in I \tag{5.31}$$

Since f is continuous on R, ϕ is continuous in I, the function F given by $F(t) = f(t, \phi(t))$ is continuous on I. Now,

$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\ dt$ and $\phi_{k+1} \to \phi(x)$ as $k \to \infty$.

Thus to prove (5.31), we must show that

$$\int_{x_0}^{x} f(t, \phi_k(t))\ dt \to \int_{x_0}^{x} f(t, \phi(t))\ dt$$

We have

$$\begin{aligned}
\left| \int_{x_0}^{x} f(t, \phi(t))\ dt - \int_{x_0}^{x} f(t, \phi_k(t))\ dt \right| &= \left| \int_{x_0}^{x} [f(t, \phi(t)) - f(t, \phi_k(t))]\ dt \right| \\
&\leq \left| \int_{x_0}^{x} |f(t, \phi(t)) - f(t, \phi_k(t))|\ dt \right| \\
&\leq \left| \int_{x_0}^{x} K\ \ |\phi(t) - \phi_k(t)|\ \ dt \right| \\
&\leq K\ \frac{M}{K}\ e^{Ka}\ \epsilon_k \left| \int_{x_0}^{x} dt \right| \\
&= M\ e^{Ka}\ \epsilon_k\ |x - x_0|
\end{aligned}$$

which tends to zzero as $k \to \infty$ for each $x \in I$. Hence $\int_{x_0}^{x} f(t, \phi_k(t))\ dt \to \int_{x_0}^{x} f(t, \phi(t))\ dt$ as $k \to \infty$. This completes the proof. □

Theorem 5.31. *The k^{th} successive approximation ϕ_k to the solution ϕ of the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ satisfies*

$$|\phi(x) - \phi_k(x)| \leq \frac{M}{K} \frac{(Ka)^{k+1}}{(k+1)\ !}\ e^{Ka} \ \textit{for all } x \textit{ in } I.$$

Exercise:

1. Consider the equation $y' = 1 - 2xy,\ y(0) = 0$ and $R:\ |x| \leq \frac{1}{2},\ |y| \leq 1$. Show that f satisfies Lipschitz condition on R with Lipschitz constant $K = 1$.

2. Consider the equation $y' = 1 + y^2,\ y(0) = 0$ and $R:\ |x| \leq \frac{1}{2},\ |y| \leq 1$. Show that f satisfies Lipschitz condition on R with Lipschitz constant $K = 1$. Also find the solution ϕ using separation of variable method. Then show that all the successive approximation ϕ_k exists and $\phi_k(x) \to \phi(x)$ for each x satisfying $|x| \leq \frac{1}{2}$.

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